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ON CERTAIN 5-MANIFOLDS WITH FUNDAMENTAL GROUP OF ORDER 2

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Abstract

In this paper, an explicit classification result for certain 5-manifolds with fundamental group $\mathbb{Z}/2$ is obtained. These manifolds include total spaces of circle bundles over simply connected 4-manifolds.

1. Introduction

The classification of manifolds with certain properties is a central topic of topology, and in dimensions ≥ 5 methods from handlebody theory and surgery have been successfully applied to a number of cases. One of the first examples was the complete classification of simply connected 5-manifolds by Smale [**21**] and Barden [**1**] in the 1960's. This result has been very useful for studying the existence of other geometric structures on 5-manifolds, such as the existence of Riemannian metrics with given curvature properties. We consider this as a model and motivation for studying the classification of non-simply connected 5-manifolds.

An orientable 5-manifold M is said to be of *fibered-type* if $\pi_2(M)$ is a trivial $\mathbb{Z}[\pi_1(M)]$-module. In this paper, we are concerned with closed, orientable fibered-type 5-manifolds M^5 with $\pi_1(M) \cong \mathbb{Z}/2$, and torsion-free $H_2(M;\mathbb{Z})$. The classification of these manifolds in the smooth (or PL) and topological categories is given in Section 3. We give a simple set of invariants, namely the rank of $H_2(M;\mathbb{Z})$ and the $\mathrm{Pin}^{\dagger}$-bordism ($\mathrm{TopPin}^{\dagger}$-bordism) class of a characteristic submanifold, which determine the diffeomorphism (homeomorphism) types. Here is the main result in the smooth case.

THEOREM 3.1 *Two smooth, closed, orientable fibered-type* 5*-manifolds M and M', with fundamental group $\mathbb{Z}/2$ and torsion-free second homology group, are diffeomorphic if and only if they have the same w_2-type*, rank $H_2(M) =$ rank $H_2(M')$ *and* $[P] = [P'] \in \Omega_4^{\mathrm{Pin}^{\dagger}}/\pm$, *where P and P' are characteristic submanifolds and $\dagger = c, -, +$ for w_2-types* I, II, III, *respectively.*

Here $\Omega_4^{\mathrm{Pin}^{\dagger}}/\pm$ denotes the quotient set of the Pin-bordism group obtained by identifying each element with its inverse (see Definition 3.5). The Pin-bordism variants and the w_2-type notation are explained in Section 2.

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The homeomorphism classification is given in Theorem 3.4. We also determine all the relations among these invariants (Theorem 3.6), and give a list of standard forms for these manifolds (Theorems 3.7 and 3.11).

One motivation for this classification problem comes from the study of circle bundles M^5 over simply connected 4-manifolds, since their total spaces are of fibered type. Duan–Liang [5] gave an explicit geometric description of M^5 for simply connected total spaces, making essential use of the results of Smale and Barden. As an application of our results, in Section 6 we give an explicit geometric description when the total spaces have fundamental group $\mathbb{Z}/2$.

THEOREM 6.5 (Type II) *Let X be a closed, simply connected, topological spin 4-manifold, and $\xi : S^1 \hookrightarrow M^5 \to X$ be a circle bundle over X with $c_1(\xi) = 2\cdot(primitive)$. Then we have the following conditions*:

(1) *if* $\mathrm{KS}(X) = 0$, *then M is smoothable and M is diffeomorphic to*

$$(S^2 \times \mathbb{R}\mathrm{P}^3)\sharp_{S^1}((\sharp_k S^2 \times S^2) \times S^1);$$

(2) *if* $\mathrm{KS}(X) = 1$, *then M is non-smoothable and M is homeomorphic to*

$$*(S^2 \times \mathbb{R}\mathrm{P}^3)\sharp_{S^1}((\sharp_k S^2 \times S^2) \times S^1),$$

where $k = \operatorname{rank} H_2(X)/2 - 1$.

In the statement, $*(S^2 \times \mathbb{R}\mathrm{P}^3)$ denotes a non-smoothable manifold homotopy equivalent to $S^2 \times \mathbb{R}\mathrm{P}^3$. The corresponding results for the other w_2-types are given in Theorems 6.7 and 6.8.

Classification results can also be useful in studying the existence problem for geometric structures on fibered-type 5-manifolds. For example, a closed, orientable 5-manifold with $\pi_1 = \mathbb{Z}/2$, such that w_2 vanishes on homology, admits a contact structure by the work of Geiges and Thomas [8]. They showed that all such manifolds can be obtained by surgery on two-dimensional links from exactly 1 of 10 model manifolds.

The topology of such manifolds of fibered-type are described explicitly for the first time by our results, and we note that all the manifolds listed in Theorem 3.7 satisfy the necessary condition $W_3 = 0$ for the existence of contact structures. Our results have already been used by Geiges and Stipsicz [7] to prove new existence theorems for contact structures on 5-manifolds. It may be possible to obtain similar information for fibered-type 5-manifolds which admit Sasakian or Einstein metrics by using the work of Boyer and Galicki [2].

The surgery exact sequence of Wall [24] provides a way to classify manifolds within a given (simple) homotopy type. However, in the application to concrete problems, one often faces homotopy theoretical difficulties. In our situation, the setting of the problems is appropriate for the application of the modified surgery methods developed by Kreck [14]. The proofs in Sections 4 and 5 are based on this theory.

In dimension 5, the smooth category and the PL category are equivalent. By convention, M stands for either a smooth or a topological manifold when not specified.

2. Preliminaries

2.1. $\mathrm{Pin}^\dagger$-*structures on vector bundles*

Recall that the groups $\mathrm{Pin}^\pm(n)$ are central extensions of $O(n)$ by $\mathbb{Z}/2$

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Pin}^\pm(n) \longrightarrow O(n) \longrightarrow 1,$$

and $\mathrm{Pin}^c(n)$ is a central extension of $O(n)$ by $U(1)$

$$1 \longrightarrow U(1) \longrightarrow \mathrm{Pin}^c(n) \longrightarrow O(n) \longrightarrow 1$$

(see [9, Section 2; 12, Section 1]). Let $\dagger \in \{c, +, -\}$. After stabilization we have classifying spaces $B\,\mathrm{Pin}^\dagger$ and fibrations $B\,\mathrm{Pin}^\dagger \to BO$. A $\mathrm{Pin}^\dagger$-structure on a stable vector bundle ξ over a space X is a fiber homotopy class of lifts of a classifying map $c_\xi : X \to BO$ to $B\,\mathrm{Pin}^\dagger$.

Lemma 2.1 [9, Lemma 1] (1) *A vector bundle ξ over X admits a $\mathrm{Pin}^\dagger$-structure if and only if*

$$\begin{aligned} \beta(w_2(\xi)) &= 0 \quad \text{for } \dagger = c,\\ w_2(\xi) &= 0 \quad \text{for } \dagger = +,\\ w_2(\xi) &= w_1(\xi)^2 \quad \text{for } \dagger = -, \end{aligned}$$

where $\beta : H^2(X; \mathbb{Z}/2) \to H^3(X; \mathbb{Z})$ is the Bockstein operator induced from the exact coefficient sequence $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2$.

(2) $\mathrm{Pin}^\pm$*-structures are in bijection with $H^1(X; \mathbb{Z}/2)$ and* Pin^c*-structures are in bijection with $H^2(X; \mathbb{Z})$.*

$\mathrm{Pin}^\pm$-structures on a vector bundle ξ over X are related to Spin-structures on an associated vector bundle.

Lemma 2.2 [12, Lemma 1.7] *Let $\mathcal{S}pin(\xi)$ denote the set of equivalence classes of Spin structures on ξ, and $\mathcal{P}in^\pm(\xi)$ denote the set of equivalence classes of* $\mathrm{Pin}^\pm$*-structures on ξ. There are bijections*

$$\begin{aligned} \mathcal{P}in^-(\xi) &\longrightarrow \mathcal{S}pin(\xi \oplus \det \xi),\\ \mathcal{P}in^+(\xi) &\longrightarrow \mathcal{S}pin(\xi \oplus 3 \det \xi), \end{aligned}$$

which are natural under the actions of $H^1(X; \mathbb{Z}/2)$.

It is well known that a Spin^c-structure on a vector bundle ξ is the same as a Spin-structure on $\xi \oplus \gamma$, where γ is a complex line bundle with $c_1(\gamma) \equiv w_2(\xi) \pmod 2$ (see [16, Corollary D.4]). Similarly, a Pin^c-structure on a vector bundle ξ may be viewed as a Pin^--structure on $\xi \oplus \gamma$, where γ is a complex line bundle with $c_1(\gamma) \equiv w_1(\xi)^2 + w_2(\xi) \pmod 2$.

2.2. *w_2-types and characteristic submanifolds*

Let M be a closed, orientable 5-manifold with $\pi_1(M) \cong \mathbb{Z}/2$ and universal cover $\widetilde{M}$. The manifold M is said to be of w_2-*type* I if $w_2(\widetilde{M}) \neq 0$, of w_2-*type* II if $w_2(M) = 0$ and of w_2-*type* III if $w_2(M) \neq 0$ and $w_2(\widetilde{M}) = 0$. By the universal coefficient theorem, there is an exact sequence

$$0 \longrightarrow \operatorname{Ext}(H_1(M), \mathbb{Z}/2) \longrightarrow H^2(M; \mathbb{Z}/2) \longrightarrow \operatorname{Hom}(H_2(M), \mathbb{Z}/2) \longrightarrow 0.$$

LEMMA 2.3 *M is of Type* III $\Leftrightarrow$ *$w_2(M) \neq 0$ and $w_2(M) \in \operatorname{Ext}(H_1(M), \mathbb{Z}/2)$.*

Proof. There is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ext}(H_1(M), \mathbb{Z}/2) & \longrightarrow & H^2(M; \mathbb{Z}/2) & \longrightarrow & \operatorname{Hom}(H_2(M), \mathbb{Z}/2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ext}(H_1(\widetilde{M}), \mathbb{Z}/2) & \longrightarrow & H^2(\widetilde{M}; \mathbb{Z}/2) & \longrightarrow & \operatorname{Hom}(H_2(\widetilde{M}), \mathbb{Z}/2) & \longrightarrow & 0
\end{array}$$

Let $p : \widetilde{M} \to M$ be the covering map; then $T\widetilde{M} = p^*TM$ and $w_2(\widetilde{M}) = p^*w_2(M)$. By the exact sequence $\pi_2(M) \to H_2(M) \to H_2(\mathbb{Z}/2) \to 0$ and the fact $H_2(\mathbb{Z}/2) = 0$ (cf. [3]), it is seen that the map $H_2(\widetilde{M}) \to H_2(M)$ is surjective; therefore the last vertical map in the diagram $\operatorname{Hom}(H_2(M), \mathbb{Z}/2) \to \operatorname{Hom}(H_2(\widetilde{M}), \mathbb{Z}/2)$ is a monomorphism. Thus, $w_2(\widetilde{M}) = 0$ if and only if $w_2(M) \in \operatorname{Ext}(H_1(M), \mathbb{Z}/2)$.

REMARK 2.4 By this lemma, the Types II and III manifolds are manifolds having second Stiefel–Whitney class equal to zero on homology. The existence of contact structures on these manifolds is shown in [8].

Recall that for a manifold M^n with fundamental group $\mathbb{Z}/2$, a *characteristic submanifold* $P^{n-1} \subset M$ is defined as follows (see [8, Section 5; 18]): there is a decomposition $\widetilde{M} = A \cup TA$ such that $\partial A = \partial(TA) = \widetilde{P}$, where T is the deck-transformation. Then $P := \widetilde{P}/T$ is called the characteristic submanifold of M. For example, if $M = \mathbb{RP}^n$, then $P = \mathbb{RP}^{n-1}$. In general, let $f : M \to \mathbb{RP}^N$ (N large) be the classifying map of the universal cover, transverse to $\mathbb{RP}^{N-1}$; then P can be taken as $f^{-1}(\mathbb{RP}^{n-1})$. By equivariant surgery, we may assume that $\pi_1(P) \cong \mathbb{Z}/2$ and that the inclusion $i : P \subset M$ induces an isomorphism on π_1. Different characteristic submanifolds of M are bordant, where a bordism is obtained from a homotopy between the relevant classifying maps. The above construction also holds in the topological category by topological transversality [11].

In the smooth category, the division of the manifolds under consideration into three w_2-types corresponds to different $\mathrm{Pin}^\dagger$-structures on their characteristic submanifolds, cf. [8, Lemma 9] for $\dagger = \pm$.

LEMMA 2.5 *Let M be a smooth, orientable 5-manifold with $\pi_1(M) \cong \mathbb{Z}/2$ and $H_2(M; \mathbb{Z})$ torsion-free. Let $P \subset M$ be a characteristic submanifold (with $\pi_1(P) \cong \pi_1(M)$). Then TP admits a* $\mathrm{Pin}^\dagger$*-structure,* *where*

$$\dagger = \begin{cases} c & \text{if } M \text{ is of Type I,} \\ - & \text{if } M \text{ is of Type II,} \\ + & \text{if } M \text{ is of Type III.} \end{cases}$$

More precisely, if M is of Type II, *then a Spin-structure on TM gives a* Pin^-*-structure on TP; if M is of Type* III, *then a Spin-structure on $TM \oplus 2L$ gives a* Pin^+*-structure on TP, where L is the non-trivial line bundle over M; if M is of Type* I, *then a Spin-structure on $TM \oplus \gamma$ gives a* Pin^c*-structure on TP, where γ is a complex line bundle over M such that $c_1(\gamma) \equiv w_2(M) \pmod 2$.*

Proof. Let $i : P \subset M$ be the inclusion and ν be the normal bundle of this inclusion; then $TP \oplus \nu = i^*TM$. If M is of Type II, a Spin-structure on TM induces a Spin-structure on $TP \oplus \nu = TP \oplus \det TP$; therefore, by Lemma 2.2, gives a Pin^--structure on TP.

If M is of Type III, then $TM \oplus 2L$ admits Spin-structures and such a structure induces a Spin-structure on $TP \oplus 3\det TP$, and henceforth a Pin^+-structure on TP.

If M is of Type I, then TP has neither Pin^- nor Pin^+-structures. Now $TM \oplus \gamma$ has Spin-structures. Such a structure induces a Spin-structure on $TP \oplus \det TP \oplus i^*\gamma$, and hence a Pin^--structure on $TP \oplus i^*\gamma$. Since $c_1(i^*\gamma) \equiv i^*w_2(M) = w_1(P)^2 + w_2(P) \pmod 2$, we obtain a Pin^c-structure on TP.

LEMMA 2.6 *If M is of Type* II *or* III, *then different characteristic submanifolds of M with the* $\mathrm{Pin}^\pm$*-structures obtained by* Lemma 2.5 *represent a pair of mutually inverse elements in the corresponding bordism group* $\Omega_4^{\mathrm{Pin}^\pm}$.

Proof. If we fix a Spin-structure on TM (or $TM \oplus 2L$), then it is clear that all different characteristic submanifolds with the induced $\mathrm{Pin}^\pm$-structure are $\mathrm{Pin}^\pm$-bordant, for they are transversal preimages of classifying maps of $\pi_1(M)$ and all such maps are homotopic. Now we fix a characteristic submanifold P, then the two $\mathrm{Pin}^\pm$-structures on TP are related by the action of $w_1(P)$, and it is a general fact that P with such two $\mathrm{Pin}^\pm$-structures give rise to a pair of mutually inverse elements in the corresponding bordism group [**12**, p. 190].

3. Main results

Now we are ready to state the classification of the manifolds under consideration.

THEOREM 3.1 *Two smooth, closed, orientable fibered-type* 5*-manifolds M and M', with fundamental group $\mathbb{Z}/2$ and torsion-free second homology group, are diffeomorphic if and only if they have the same w_2-type,* $\operatorname{rank} H_2(M) = \operatorname{rank} H_2(M')$ *and* $[P] = [P'] \in \Omega_4^{\mathrm{Pin}^\dagger}/\pm$, *where P and P' are the characteristic submanifolds and $\dagger = c, -, +$ for Types* I, II, III, *respectively.*

REMARK 3.2 The notation $\Omega_4^{\mathrm{Pin}^\dagger}/\pm$ was explained in Section 1. It is known that $\Omega_4^{\mathrm{Pin}^-} = 0$ [**12**]. Therefore, $\operatorname{rank} H_2(M)$ is the only diffeomorphism invariant for the Type II manifolds.

There are topological versions of the central extensions mentioned above and we have groups $\mathrm{TopPin}^\dagger(n)$, $\dagger \in \{c, +, -\}$. For the preliminaries on $\mathrm{TopPin}^\dagger(n)$ we refer to [**9**, **12**]. Therefore, we have corresponding results in the topological category.

LEMMA 3.3 *Let M be a topological, orientable 5-manifold with $\pi_1(M) \cong \mathbb{Z}/2$ and $H_2(M;\mathbb{Z})$ torsion-free. Let $P \subset M$ be a characteristic submanifold (with $\pi_1(P) \cong \pi_1(M)$). Then TP admits a* $\mathrm{TopPin}^{\dagger}$-*structure, where*

$$\dagger = \begin{cases} c & \text{if } M \text{ is of Type I,} \\ - & \text{if } M \text{ is of Type II,} \\ + & \text{if } M \text{ is of Type III.} \end{cases}$$

THEOREM 3.4 *Two topological, closed, orientable fibered-type 5-manifolds M and M', with fundamental group $\mathbb{Z}/2$ and torsion-free second homology group, are homeomorphic if and only if they have the same w_2-type,* $\operatorname{rank} H_2(M) = \operatorname{rank} H_2(M')$ *and $[P] = [P'] \in \Omega_4^{\mathrm{TopPin}^{\dagger}}/\pm$, where P and P' are characteristic submanifolds and $\dagger = c, -, +$ for Types* I, II, III, *respectively.*

The groups $\Omega_4^{\mathrm{Pin}^{\pm}}$ and $\Omega_4^{\mathrm{TopPin}^{\pm}}$ are computed in [12]. $\Omega_4^{\mathrm{TopPin}^{c}}$ is computed in [9, p. 654]. (Note that the rôle of Pin^{+} and Pin^{-} in [9] are reversed since in that paper the authors consider normal structures, whereas here we use the convention in [12], looking at the tangential Gauss map.) In a similar way, we shall compute $\Omega_4^{\mathrm{Pin}^{c}}$ below. We list the values of these groups:

$\dagger$	$\Omega_4^{\mathrm{Pin}^{\dagger}}$	Invariants	Generators
c	$\mathbb{Z}/8 \oplus \mathbb{Z}/2$	(arf, w_2^2)	$\mathbb{RP}^4$, $\mathbb{CP}^2$
$+$	$\mathbb{Z}/16$	?	$\mathbb{RP}^4$
$-$	0	—	—
$\dagger$	$\Omega_4^{\mathrm{TopPin}^{\dagger}}$	Invariants	Generators
c	$\mathbb{Z}/2 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2$	(KS, arf, w_2^2)	E_8, $\mathbb{RP}^4$, $\mathbb{CP}^2$
$+$	$\mathbb{Z}/2 \oplus \mathbb{Z}/8$	(KS, arf)	E_8, $\mathbb{RP}^4$
$-$	$\mathbb{Z}/2$	KS	E_8

Computation of $\Omega_4^{\mathrm{Pin}^{c}}$: the extension

$$1 \longrightarrow \mathrm{Pin}^{-} \longrightarrow \mathrm{Pin}^{c} \longrightarrow U(1) \longrightarrow 1$$

induces Gysin sequence (cf. [9, p. 654])

$$\cdots \longrightarrow \Omega_4^{\mathrm{Pin}^{-}} \longrightarrow \Omega_4^{\mathrm{Pin}^{c}} \xrightarrow{\cap c} \Omega_2^{\mathrm{Pin}^{-}}(BU(1)) \longrightarrow \Omega_3^{\mathrm{Pin}^{-}} \longrightarrow \cdots .$$

Since $\Omega_4^{\mathrm{Pin}^{-}} = \Omega_3^{\mathrm{Pin}^{-}} = 0$ (see [12]), we have an isomorphism

$$\Omega_4^{\mathrm{Pin}^{c}} \xrightarrow{\cap c} \Omega_2^{\mathrm{Pin}^{-}}(BU(1)),$$

and the latter group is the same as $\Omega_2^{\mathrm{TopPin}^{-}}(BU(1))$, which is computed in [9].

The invariants in Theorem 3.1 are subject to certain relations.

DEFINITION 3.5 *Define* $r = \operatorname{rank} H_2(M)$, $q = [P] \in \Omega_4^{\mathrm{Pin}^{+}}/\pm = \{0, 1, \dots, 8\}$ *and* $(q, s) = [P] \in \Omega_4^{\mathrm{Pin}^{c}}/\pm = \{0, 1, \dots, 4\} \times \{0, 1\}$.

As an application of the semi-characteristic class [17], we have the following theorem.

THEOREM 3.6 *Let M be a smooth, orientable 5-manifold with $\pi_1(M) \cong \mathbb{Z}/2$ and torsion-free $H_2(M)$, having the invariants as above. Then these invariants are subject to the following relations*:

Type	*Relation*
I	$q + s + r \equiv 1 \pmod 2$
II	$r \equiv 1 \pmod 2$
III	$q + r \equiv 1 \pmod 2$

Now we give a list of all the manifolds under consideration, realizing the possible invariants. We need some preliminaries.

By a computation of the surgery exact sequence, it is shown in [24] that in the smooth (or PL) category, there are four distinct diffeomorphism types of manifolds which are homotopy equivalent to $\mathbb{RP}^5$; these are called fake $\mathbb{RP}^5$. An explicit construction using links of singularities (Brieskorn spheres) can be found in [8]. Following the notation there, we denote these fake $\mathbb{RP}^5$ by $X^5(q)$, $q = 1, 3, 5, 7$, with $X^5(1) = \mathbb{RP}^5$. These manifolds fall into the class of manifolds under consideration. They are of Type III and the Pin^+-bordism class of the corresponding characteristic submanifold is $q \in \Omega_4^{\mathrm{Pin}^+}/\pm = \{0, 1, \ldots, 8\}$; see [8]. In our list of standard forms, these fake projective spaces will serve as building blocks under the operation $\sharp_{S^1}$—'connected-sum along S^1', which we explain now; cf. [9].

Connected sum along a circle. Let M_i $(i = 1, 2)$ be oriented 5-manifolds with fundamental group $\mathbb{Z}/2$ or $\mathbb{Z}$, and at least one of the fundamental groups is $\mathbb{Z}/2$. Denote the trivial oriented four-dimensional real disk bundle over S^1 by E. Choose embeddings of E into M_1 and M_2, representing a generator of $\pi_1(M_i)$, such that the first embedding preserves the orientation and the second reverses it. Then we define

$$M_1 \sharp_{S^1} M_2 := (M_1 - E) \cup_\partial (M_2 - E).$$

Note that if one of the 5-manifolds admits an orientation-reversing automorphism, then the construction does not depend on the orientations, and this is the case for the building blocks in the list below, namely, $S^2 \times \mathbb{RP}^3$, $S^2 \times S^2 \times S^1$, $X^5(q)$ and $\mathbb{CP}^2 \times S^1$ admit orientation-reversing automorphisms. (The fact that $X^5(q)$ admits orientation-reversing automorphisms follows from the fact that $\mathbb{RP}^5$ admits orientation-reversing automorphisms and that the action of $\mathrm{Aut}(\mathbb{RP}^5)$ on the structure set $\mathscr{S}(\mathbb{RP}^5)$ is trivial.)

The Seifert–van Kampen theorem implies that $\pi_1(M_1 \sharp_{S^1} M_2) \cong \mathbb{Z}/2$. The Mayer–Vietoris exact sequence implies that $H_2(M_1 \sharp_{S^1} M_2)$ is torsion-free, and hence $M_1 \sharp_{S^1} M_2$ is of fibered type. The homology rank $H_2(M_1 \sharp_{S^1} M_2) = \operatorname{rank} H_2(M_1) + \operatorname{rank} H_2(M_2) + 1$ if both fundamental groups are $\mathbb{Z}/2$, and rank $H_2(M_1 \sharp_{S^1} M_2) = \operatorname{rank} H_2(M_1) + \operatorname{rank} H_2(M_2)$ if one of the fundamental groups is $\mathbb{Z}$.

Since $\pi_1 \mathrm{SO}(4) \cong \mathbb{Z}/2$, there are actually two possibilities to form $M_1 \sharp_{S^1} M_2$. However, from the classification result, it turns out that this ambiguity happens only when we construct $X^5(q) \sharp_{S^1} X^5(q')$. This does depend on the framings, and therefore $X^5(q) \sharp_{S^1} X^5(q')$ represents 2-manifolds. Note that the characteristic submanifold of $M_1 \sharp_{S^1} M_2$ is $P_1 \sharp_{S^1} P_2$ (see [9, p. 651] for the definition of $\sharp_{S^1}$ for non-orientable 4-manifolds with fundamental group $\mathbb{Z}/2$). Therefore, if we fix Pin^+-structures on each of the characteristic submanifolds, then $X^5(q) \sharp_{S^1} X^5(q')$ is well defined.

This construction allows us to construct manifolds with a given bordism class of characteristic submanifold. Note that $P_1 \sharp_{S^1} P_2$ corresponds to the addition in the bordism group $\Omega_4^{\mathrm{Pin}^\dagger}$. Now, for

$q = 0, 2, 4, 6, 8$, choose $l, l' \in \{1, 3, 5, 7\}$ and appropriate Pin^+-structures on the characteristic submanifolds of $X^5(l)$ and $X^5(l')$, we can form a manifold $X^5(l)\sharp_{S^1} X^5(l')$ such that the characteristic submanifold $[P] = q \in \Omega_4^{\mathrm{Pin}^+}/\pm$. We denote this manifold also by $X^5(q)$. For example, we can form $X^5(0) = X^5(1)\sharp_{S^1} X^5(1)$ and $X^5(2) = X^5(1)\sharp_{S^1} X^5(1)$ with different glueing maps.

With these notations, the list of standard forms of the manifolds under consideration is given as follows.

THEOREM 3.7 *Every closed smooth orientable fibered-type 5-manifold with fundamental group $\mathbb{Z}/2$ and second homology group $\mathbb{Z}^r$ is diffeomorphic to exactly one of the following standard forms*:

Type I: $X^5(q)\sharp_{S^1}(S^2 \times \mathbb{R}\mathrm{P}^3)\sharp_{S^1}((\sharp_k S^2 \times S^2) \times S^1)$, $\quad r = 2k + (5 + (-1)^q)/2$, $q \in \{0, \dots, 4\}$;
$X^5(q)\sharp_{S^1}(\mathbb{C}\mathrm{P}^2 \times S^1)\sharp_{S^1}((\sharp_k S^2 \times S^2) \times S^1)$, $\quad r = 2k + (3 + (-1)^q)/2$, $q \in \{0, \dots, 4\}$;
Type II: $(S^2 \times \mathbb{R}\mathrm{P}^3)\sharp_{S^1}((\sharp_k S^2 \times S^2) \times S^1)$, $\quad r = 2k + 1$;
Type III: $X^5(q)\sharp_{S^1}((\sharp_k S^2 \times S^2) \times S^1)$, $\quad r = 2k + (1 + (-1)^q)/2$, $q \in \{0, \dots, 8\}$,

where $\sharp_k S^2 \times S^2$ is the connected sum of k copies of $S^2 \times S^2$.

REMARK 3.8 There can be other descriptions of the manifolds in the list. For example, we have a (more symmetric) description of the Type II standard forms

$$\underbrace{(S^2 \times \mathbb{R}\mathrm{P}^3)\sharp_{S^1} \cdots \sharp_{S^1}(S^2 \times \mathbb{R}\mathrm{P}^3)}_{k \text{ times}}.$$

REMARK 3.9 Note that the universal covers of the manifolds under consideration have torsion-free second homology, and therefore, according to the results of Smale and Barden, are diffeomorphic to $\sharp_r(S^2 \times S^3)$ or $B\sharp_{r-1}(S^2 \times S^3)$, where B is the non-trivial S^3-bundle over S^2. From this point of view, Theorem 3.7 gives the classification of orientation-preserving free involutions on $\sharp_r(S^2 \times S^3)$ and $B\sharp_{r-1}(S^2 \times S^3)$, which act trivially on H_2. For example, consider the orientation-preserving free involution on $S^2 \times S^3$ given by $(x, y) \mapsto (r(x), -y)$, where $r : S^2 \to S^2$ is the reflection along a line and $- : S^3 \to S^3$ is the antipodal map. Then the quotient space is actually the sphere bundle of the non-trivial orientable $\mathbb{R}^3$-bundle over $\mathbb{R}\mathrm{P}^3$. From Theorem 3.1 it is easy to see that this is just $X^5(0)$.

REMARK 3.10 The above list may be of use in the study of geometric structures on these manifolds. Geiges and Thomas [8] show that the Types II and III manifolds admit contact structures. On the other hand, a necessary condition for the existence of contact structures on M^{2n+1} is the reduction of the structure group of TM to $U(n)$, hence the vanishing of integral Stiefel–Whitney classes $W_{2i+1}(M)$. It is easy to see that the Type I manifolds satisfy this necessary condition. These manifolds also satisfy the necessary conditions on the cup length and Betti numbers in [2] for the existence of Sasakian structures. Therefore, it would be interesting to study these geometric structures on these manifolds.

Proof of Theorem 3.7. By the Van-Kampen theorem and the Mayer–Vietoris sequence, it is easy to see that all the manifolds in the list are orientable, with fundamental group $\mathbb{Z}/2$ and torsion-free H_2, and the π_1-action on H_2 is trivial. Therefore, we only need to verify that these manifolds have different invariants and realize all the possible invariants.

Type II: rank $H_2((S^2 \times \mathbb{R}\mathrm{P}^3)\sharp_{S^1}((\sharp_k S^2 \times S^2) \times S^1)) = 2k + 1$.

Type III: the characteristic submanifold of $X^5(q)\sharp_{S^1}((\sharp_k S^2 \times S^2) \times S^1)$ is just that of $X^5(q)$, which corresponds to $q \in \Omega_4^{\mathrm{Pin}^+}/\pm = \{0, \ldots, 8\}$.

Type I: similarly, the manifold $X^5(q)\sharp_{S^1}(\mathbb{C}\mathrm{P}^2 \times S^1)\sharp_{S^1}((\sharp_k S^2 \times S^2) \times S^1)$ has characteristic submanifold invariant $(q, 1) \in \Omega_4^{\mathrm{Pin}^c}/\pm$.

To give a list of standard forms of the manifolds under consideration in the topological case, we need a topological 5-manifold which is homotopy equivalent to $S^2 \times \mathbb{R}\mathrm{P}^3$ and whose characteristic submanifold represents the non-trivial element in $\Omega_4^{\mathrm{TopPin}^-} = \mathbb{Z}/2$. Note that by Theorem 3.4, if such manifolds exist, then the homeomorphism type is unique. Following the notation in [9], we denote this manifold by $*(S^2 \times \mathbb{R}\mathrm{P}^3)$. We now give the construction of $*(S^2 \times \mathbb{R}\mathrm{P}^3)$.

Let $W = S^2 \times \mathbb{R}\mathrm{P}^3\sharp_{S^1} E_8 \times S^1$, so that $\pi_1(W) = \mathbb{Z}/2$ and the characteristic submanifold of W is $S^2 \times \mathbb{R}\mathrm{P}^2\sharp E_8$. Let $h : W \to S^2 \times \mathbb{R}\mathrm{P}^3$ be a degree 1 normal map which extends the degree 1 normal map $f : S^2 \times \mathbb{R}\mathrm{P}^2\sharp E_8 \to S^2 \times \mathbb{R}\mathrm{P}^2$. Then, by doing codimension 1 surgery on h, we obtain a W' with characteristic submanifold $P = *(S^2 \times \mathbb{R}\mathrm{P}^2)$ and a degree 1 normal map $h' : W' \to S^2 \times \mathbb{R}\mathrm{P}^3$ extending a homotopy equivalence $f' : *(S^2 \times \mathbb{R}\mathrm{P}^2) \to S^2 \times \mathbb{R}\mathrm{P}^2$ (cf. [9] for the construction of $*(S^2 \times \mathbb{R}\mathrm{P}^2)$). The π-π theorem allows us to do further surgeries on the complement of a tubular neighborhood of P to obtain a homotopy equivalence.

In the topological category, there are four fake $\mathbb{R}\mathrm{P}^5$'s. Two of them are smoothable. We denote these manifolds by $X^5(p, q)$ ($p \in \{0, 1\}$, $q \in \{1, 3\}$) such that the characteristic submanifold of $X^5(p, q)$ is $(p, q) \in \Omega_4^{\mathrm{TopPin}^+}/\pm = \{0, 1\} \times \{0, 1, 2, 3, 4\}$. Similar to the smooth case, we can also construct $X^5(p, q)$ ($p \in \{0, 1\}$, $q \in \{0, 2, 4\}$) by a (?) circle connected sum of fake $\mathbb{R}\mathrm{P}^5$. (Note that the Kirby–Siebenmann invariant is additive under the connected sum operation [20].)

THEOREM 3.11 *Every closed topological orientable fibered-type* 5*-manifold with fundamental group* $\mathbb{Z}/2$ *and second homology group* $\mathbb{Z}^r$ *is homeomorphic to exactly one of the following standard forms*:

Type I: $X^5(p, q)\sharp_{S^1}(S^2 \times \mathbb{R}\mathrm{P}^3)\sharp_{S^1}((\sharp k\, S^2 \times S^2) \times S^1)$,
$r = 2k + (5 + (-1)^q)/2,\ q \in \{0, \ldots, 4\},\ p = 0, 1;$
$X^5(p, q)\sharp_{S^1}(\mathbb{C}\mathrm{P}^2 \times S^1)\sharp_{S^1}((\sharp k\, S^2 \times S^2) \times S^1)$,
$r = 2k + (3 + (-1)^q)/2,\ q \in \{0, \ldots, 4\},\ p = 0, 1;$

Type II: $(S^2 \times \mathbb{R}\mathrm{P}^3)\sharp_{S^1}((\sharp k\, S^2 \times S^2) \times S^1)$, $r = 2k + 1$;
$*(S^2 \times \mathbb{R}\mathrm{P}^3)\sharp_{S^1}((\sharp k\, S^2 \times S^2) \times S^1)$, $r = 2k + 1$;

Type III: $X^5(p, q)\sharp_{S^1}((\sharp k\, S^2 \times S^2) \times S^1)$, $r = 2k + (1 + (-1)^q)/2,\ q \in \{0, \ldots, 4\},\ p = 0, 1.$

From the above list, we can also give a homotopy classification.

THEOREM 3.12 *The homotopy type of* M^5 *is determined by its* w_2*-type*, rank $H_2(M)$, *and in the Type* I *case the number* $\langle w_2(M)^2 \cup t + t^5, [M]\rangle \in \mathbb{Z}/2$, *where* $t \in H^1(M; \mathbb{Z}/2)$ *is the non-zero element.*

Proof. Note that $X^5(q)$ and $X^5(p, q)$ are homotopy equivalent to $\mathbb{R}\mathrm{P}^5$ and the operation $\sharp_{S^1}$ preserves homotopy equivalence. This proves the theorem for the Types II and III cases. For Type I manifolds the s-component of the characteristic submanifold P is determined by $\langle w_2(P)^2, [P]\rangle$. Since $w_2(P) = i^*(w_2(M) + t^2)$, it follows that $\langle w_2(P)^2, [P]\rangle = \langle w_2(M)^2 \cup t + t^5, [M]\rangle$, and this is a homotopy invariant.

4. Bordism and surgery

4.1. *The framework of modified surgery*

The main tool used in our solution of the classification problem is the modified surgery developed by Kreck [13, 14]. We first briefly describe how this theory is applied in our situation.

Let $p : B \to BO$ be a fibration, and $\bar{\nu} : M^{2m-1} \to B$ be a lift of the normal Gauss map $\nu : M \to BO$ classifying the stable normal bundle of M. Such a lift $\bar{\nu}$ is called a *normal B-structure* of M, and the pair $(M, \bar{\nu})$ is called a *normal k-smoothing in B* if the map $\bar{\nu}$ is a $(k+1)$-equivalence. Manifolds with normal B-structures form a bordism theory $\Omega_*(B, p)$, described in Stong [22, Chapter II].

Suppose $(M_i^{2m-1}, \bar{\nu}_i)$ $(i = 1, 2)$ are two normal $(m-1)$-smoothings in B, and suppose that $(W^{2m}, \bar{\nu})$ is a B-bordism between $(M_1^{2m-1}, \bar{\nu}_1)$ and $(M_2^{2m-1}, \bar{\nu}_2)$. Then the surgery obstruction for W^{2m} being B-bordant relative to the boundary to an s-cobordism (implying that M_1 and M_2 are diffeomorphic) is a $(-1)^m$-quadratic form over (Λ, S), where $\Lambda = \mathbb{Z}[\pi_1(B)]$ is the group ring and $S \subset \Lambda$ is a certain form-parameter subgroup. The surgery obstruction lies in an abelian group $L_{2m}^{s,\tau}(\pi_1(B), w_1(B), S)$ [13, Theorem 5.2 b], where $w_1(B)$ is the orientation character. This group is related to Wall's L-group in the following diagram [13, p. 37]:

$$\begin{tikzcd}
0 \arrow[r] & L_{2m}^{s}(\pi_1, w_1) \arrow[r] & L_{2m}^{s,\tau}(\pi_1, w_1) \arrow[r] \arrow[d] & \mathrm{Wh}(\pi_1) \\
& & L_{2m}^{s,\tau}(\pi_1, w_1, S) \arrow[d] & \\
& & 0 &
\end{tikzcd}$$

where $\mathrm{Wh}(\pi_1)$ is the Whitehead group (see [19]).

In our case, $\pi_1 = \mathbb{Z}/2$, $\mathrm{Wh}(\mathbb{Z}/2) = 0$ and $L_6^s(\mathbb{Z}/2) = \mathbb{Z}/2$. Therefore, our surgery obstruction group is either 0 or $\mathbb{Z}/2$. In the latter case, it is isomorphic to $L_6^s(\mathbb{Z}/2)$, the non-trivial element is detected by the Kervaire–Arf invariant (see Wall [24, Section 13A]). Since the closed manifold $S^3 \times S^3$ admits a framing with Arf invariant 1, we may eliminate the surgery obstruction by a connected sum in the interior of W. We have the following proposition.

PROPOSITION 4.1 *Two smooth 5-manifolds M_1 and M_2 with fundamental group $\mathbb{Z}/2$ are diffeomorphic if they have bordant normal 2-smoothings in some fibration B.*

The fibration B is called the *normal 2-type of M* if p is 3-coconnected. This is an invariant of M. Because of this proposition, the solution to the classification problem consists of two steps: first, determine the normal 2-types B for the 5-manifolds under consideration, and then determine invariants to detect the corresponding bordism groups $\Omega_5(B, p)$.

4.2. *Normal 2-types*

Let M^5 be a fibered-type 5-manifold. The universal coefficient theorem implies that $H_2(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1]} \mathbb{Z} \to H_2(M)$ is an isomorphism. Since the $\pi_1(M)$-action on $H_2(\tilde{M})$ is trivial, we have $H_2(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1]}$

$\mathbb{Z} = H_2(\tilde{M})$, therefore $H_2(\tilde{M}) \to H_2(M)$ is an isomorphism, and also is the second Hurewicz map $\pi_2(M) \to H_2(M)$. Now suppose $\pi_1(M) \cong \mathbb{Z}/2$ and $H_2(M) \cong \mathbb{Z}^r$.

We start with the description of the normal 2-types for Type II manifolds. It is the simplest situation and illuminates the ideas.

Type II: consider the fibration

$$p : B = \mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r \times B\,\mathrm{Spin} \longrightarrow BO,$$

where $p : B \to BO$ is trivial on the first two factors and on $B\,\mathrm{Spin}$ it is the canonical projection from $B\,\mathrm{Spin}$ onto BO. A lift $\bar{\nu} : M \to B$ is given as follows: the map to $\mathbb{R}\mathrm{P}^\infty$ is the classifying map of the fundamental group; choose a basis $\{u_1, \dots, u_r\}$ of the free part of $H^2(M) \cong \mathbb{Z}^r \oplus \mathbb{Z}/2$, and by realizing each element u_i by a map to $\mathbb{C}\mathrm{P}^\infty$ we get a map to $(\mathbb{C}\mathrm{P}^\infty)^r$; a Spin-structure on νM gives rise to a map to $B\,\mathrm{Spin}$. It is easy to see that (B, p) is the normal 2-type of Type II manifolds and that $\bar{\nu}$ induces an isomorphism on π_1 and H_2. Since the second Hurewicz maps $\pi_2(M) \to H_2(M)$ and $\pi_2((\mathbb{C}\mathrm{P}^\infty)^r) \to H_2((\mathbb{C}\mathrm{P}^\infty)^r)$ are isomorphisms, $\bar{\nu}$ is a normal 2-smoothing.

Type III: let η be the canonical real line bundle over $\mathbb{R}\mathrm{P}^\infty$, and $2\eta = \eta \oplus \eta$. Consider the fibration

$$p : B = \mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r \times B\,\mathrm{Spin} \xrightarrow{f_1 \times f_2} BO \times BO \xrightarrow{\oplus} BO,$$

where $f_1 : \mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r \to BO$ is the classifying map of $p_1^*(2\eta)$ (where $p_1 : \mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r \to \mathbb{R}\mathrm{P}^\infty$ is the projection map), $f_2 : B\,\mathrm{Spin} \to BO$ is the canonical projection and $\oplus : BO \times BO \to BO$ is the H-space structure on BO induced by the Whitney sum of vector bundles. A lift $\bar{\nu} : M \to B$ is given as follows: the map, to $\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r$ is the same as in Type II. Since $w_2(2\eta) = w_1(\eta)^2$ is the non-zero element in $\mathrm{Ext}(H_1(\mathbb{R}\mathrm{P}^\infty), \mathbb{Z}/2)$ and $w_2(M)$ is the non-zero element in $\mathrm{Ext}(H_1(M), \mathbb{Z}/2)$, we have $w_2(\bar{\nu}^* 2\eta) = w_2(\nu M)$. This implies that $\nu M - \bar{\nu}^* 2\eta$ admits a Spin-structure. Such a structure induces a map to $B\,\mathrm{Spin}$. Then $\bar{\nu}$ is a lift of ν. It is easy to see that (B, p) is the normal 2-type of Type III manifolds and $\bar{\nu}$ is a normal 2-smoothing.

Type I: let γ be the canonical complex line bundle over $\mathbb{C}\mathrm{P}^\infty$. Consider the fibration

$$p : B = \mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r \times B\,\mathrm{Spin} \xrightarrow{f_1 \times f_2} BO \times BO \xrightarrow{\oplus} BO,$$

where $f_1 : \mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r \to BO$ is the classifying map of $p_2^* \gamma$ and $p_2 : \mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r \to \mathbb{C}\mathrm{P}^\infty$ is the projection map to the first $\mathbb{C}\mathrm{P}^\infty$. A lift $\bar{\nu} : M \to B$ is given as follows: since the Bockstein homomorphism $\beta : H^2(M; \mathbb{Z}/2) \to H^3(M; \mathbb{Z})$ is trivial, $w_2(M)$ is the mod 2 reduction of an integral cohomology class. Since $w_2(M)$ is not contained in $\mathrm{Ext}(H_1(M), \mathbb{Z}/2)$, this integral cohomology class can be taken as a primitive one, say, u_1 and we extend it to a basis $\{u_1, \dots, u_r\}$. Then the map to $\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r$ is the same as above. Now $\nu M - \bar{\nu}^* \gamma$ admits a Spin-structure; this gives rise to a map $M \to B\,\mathrm{Spin}$. Then $\bar{\nu}$ is a lift of ν. It is easy to see that (B, p) is the normal 2-type of Type I manifolds and $\bar{\nu}$ is a normal 2-smoothing.

4.3. *Computation of the bordism groups*

In this subsection, we calculate the bordism groups $\Omega_5(B, p)$ for our types:

$$\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r), \quad \Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r; p_1^* 2\eta), \quad \Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r; p_2^* \gamma).$$

The main tools are the Atiyah–Hirzebruch spectral sequence and the Adams spectral sequence. Before doing the calculation, we need to compute $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty)$, $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty; 2\eta)$ and $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; p_2^*\gamma)$. These groups can be calculated via the Adams spectral sequence. Here we give an alternative argument, emphasizing the role of the characteristic submanifolds.

There are long exact sequences (this is a special case of Galatius *et al.* [6, (3.2)])

$$\cdots \to \Omega_n^{\mathrm{Spin}} \longrightarrow \Omega_n^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty; k\eta) \xrightarrow{\partial} \Omega_{n-1}^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty; (k+1)\eta) \longrightarrow \Omega_{n-1}^{\mathrm{Spin}} \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \Omega_n^{\mathrm{Spin}}(\mathbb{C}\mathrm{P}^\infty; \gamma) \longrightarrow \Omega_n^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; p_2^*\gamma) \xrightarrow{\partial} \Omega_{n-1}^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \eta \times \gamma) \longrightarrow \cdots,$$

where the maps ∂ correspond to taking a characteristic submanifold. In particular, we have an isomorphism $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty) \xrightarrow{\cong} \Omega_4^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty; \eta)$, together with exact sequences

$$0 \longrightarrow \Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty; 2\eta) \longrightarrow \Omega_4^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty; 3\eta)$$

and

$$\Omega_5^{\mathrm{Spin}}(\mathbb{C}\mathrm{P}^\infty; \gamma) \longrightarrow \Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; p_2^*\gamma) \longrightarrow \Omega_4^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \eta \times \gamma)$$
$$\longrightarrow \Omega_4^{\mathrm{Spin}}(\mathbb{C}\mathrm{P}^\infty; \gamma).$$

Furthermore, we have

$$\Omega_n^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty; \eta) \cong \Omega_n^{\mathrm{Pin}^-}, \quad \Omega_n^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty; 3\eta) \cong \Omega_n^{\mathrm{Pin}^+}, \quad \Omega_n^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \eta \times \gamma) \cong \Omega_n^{\mathrm{Pin}^c}.$$

This is seen as follows: first, given $[X^n, f] \in \Omega_n^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty; \eta)$, clearly

$$w_1(f^*\eta) = w_1(X) = w_1(\det TX).$$

Therefore, by Lemma 2.2, the Spin-structure on $TX \oplus f^*\eta$ induces a Pin^--structure on TX and we have a well-defined map $\Omega_n^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty; \eta) \to \Omega_n^{\mathrm{Pin}^-}$. Given X^n together with a Pin^--structure, by letting $f : X \to \mathbb{R}\mathrm{P}^\infty$ be the classifying map for $w_1(X)$, we obtain $[X, f] \in \Omega_n^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty; \eta)$. These two maps are inverse to each other. The Pin^+ and Pin^c cases are similar.

The $\mathrm{Pin}^\pm$-bordism groups in low dimensions were calculated in [12]: we have $\Omega_4^{\mathrm{Pin}^-} = 0$ and $\Omega_4^{\mathrm{Pin}^+} \cong \mathbb{Z}/16$, generated by $\pm\mathbb{R}\mathrm{P}^4$. Also it is clear that, under the map

$$\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty; 2\eta) \longrightarrow \Omega_4^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty; 3\eta) \cong \Omega_4^{\mathrm{Pin}^+},$$

the element $[\mathbb{R}\mathrm{P}^5, \text{inclusion}]$ goes to $\pm\mathbb{R}\mathrm{P}^4$, therefore the map

$$\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty; 2\eta) \to \Omega_4^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty; 3\eta)$$

is an isomorphism. An easy Atiyah–Hirzebruch spectral sequence calculation shows that $\Omega_5^{\mathrm{Spin}}(\mathbb{C}\mathrm{P}^\infty; \gamma) \cong \widetilde{\Omega}_7^{\mathrm{Spin}}(\mathbb{C}\mathrm{P}^\infty) = 0$ and $\Omega_4^{\mathrm{Spin}}(\mathbb{C}\mathrm{P}^\infty; \gamma) \cong \widetilde{\Omega}_6^{\mathrm{Spin}}(\mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z} \oplus \mathbb{Z}$. Therefore, the map $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; p_2^*\gamma) \to \Omega_4^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \eta \times \gamma)$ is also an isomorphism. To summarize, we have the following lemma.

LEMMA 4.2 *Taking characteristic submanifolds gives isomorphisms*

$$\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty) \cong \Omega_4^{\mathrm{Pin}^-}, \quad \Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty; 2\eta) \cong \Omega_4^{\mathrm{Pin}^+}, \quad \Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; p_2^*\gamma) \cong \Omega_4^{\mathrm{Pin}^c}.$$

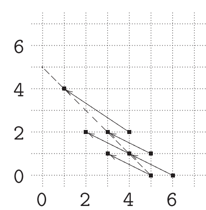
Now we begin the calculation of the bordism groups of interest. As in the last subsection, we start with the Type II manifolds, which is the simplest case.

Type II: recall that the normal 2-type is

$$p : B = \mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r \times B\,\mathrm{Spin} \longrightarrow BO,$$

where $p : B \to BO$ is trivial on the first two factors and is the canonical projection from $B\,\mathrm{Spin}$ onto BO. Therefore, the bordism group $\Omega_5(B, p)$ is the Spin-bordism group $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r)$. To compute this bordism group, we apply the Atiyah–Hirzebruch spectral sequence. The E^2-terms are $E^2_{p,q} = H_p(\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r; \Omega_q^{\mathrm{Spin}})$.

To illuminate the situation, we first consider the group $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty)$. The relevant terms and differentials in the spectral sequence are depicted as follows:



The E^2-terms are:

(1) $E^2_{1,4} = H_1(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}/2$,
(2) $E^2_{2,2} = H_2(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$,
(3) $E^2_{3,1} = E^2_{3,2} = H_3(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$,
(4) $E^2_{4,1} = E^2_{4,2} = H_4(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^3$,
(5) $E^2_{5,0} = H_5(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong (\mathbb{Z}/2)^3$,
(6) $E^2_{5,1} = H_5(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^3$,
(7) $E^2_{6,0} = H_6(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}/2$.

The differential $d_2 : E^2_{p,1} \to E^2_{p-2,2}$ is dual to the Steenrod square

$$\mathrm{Sq}^2 : H^{p-2}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2) \longrightarrow H^p(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2);$$

the differential $d_2 : E^2_{p,0} \to E^2_{p-2,1}$ is the mod 2 reduction composed with the dual of the Steenrod square

$$H_p(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}) \longrightarrow H_p(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2) \xrightarrow{(\mathrm{Sq}^2)^*} H_{p-2}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2).$$

With these identifications, the differentials d_2 starting from or ending at the line $p + q = 5$ are easily computed. Let $\alpha \in H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2)$, $\beta \in H^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2)$ denote the generators; then, on the E^3-page, we have three non-trivial terms in the line $p+q=5$: $E^3_{5,0}=\mathbb{Z}/2$, dual to $\alpha^3\beta$; $E^3_{4,1}=\mathbb{Z}/2$, dual to $\alpha^2\beta$ and $E^3_{1,4}=\mathbb{Z}/2$. The terms $E^3_{5,0}$ and $E^3_{4,1}$ must survive to infinity, for there are no non-trivial differentials starting from or ending at these two positions as depicted in the above inline diagram.

There is a possibly non-trivial differential $d_3 : E^3_{4,2} \to E^3_{1,4}$. To see that this differential is indeed non-trivial, we just need to note that the terms $E^3_{1,4}=E^2_{1,4}=H_1(\mathbb{RP}^\infty \times \mathbb{CP}^\infty;\mathbb{Z}/2)$ come from $\mathbb{RP}^\infty$ and $\Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty)=\Omega_4^{\mathrm{Pin}^-}=0$. Therefore, on the E^∞-page, in the line $p+q=5$, the non-trivial terms are

$$E^\infty_{5,0}=H_3(\mathbb{RP}^\infty)\otimes H_2(\mathbb{CP}^\infty)\cong \mathbb{Z}/2, \quad E^\infty_{4,1}=H_2(\mathbb{RP}^\infty;\mathbb{Z}/2)\otimes H_2(\mathbb{CP}^\infty;\mathbb{Z}/2)\cong \mathbb{Z}/2.$$

The calculation is completed once the extension problem is solved. We state the result in the following lemma. Let $\tau : \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ be the involution on $\mathbb{CP}^\infty$ with $\tau_* = -1$ on $H_2(\mathbb{CP}^\infty)$; then τ induces an involution τ_* on $\Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty\times\mathbb{CP}^\infty)$. Let $\alpha \in H^1(\mathbb{RP}^\infty;\mathbb{Z}/2)$, $\beta \in H^2(\mathbb{CP}^\infty;\mathbb{Z}/2)$ be the non-zero elements.

LEMMA 4.3 *The short exact sequence*

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty\times\mathbb{CP}^\infty) \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

is non-split, thus $\Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty\times\mathbb{CP}^\infty)\cong\mathbb{Z}/4$. *The elements* ±1 *are represented by* $\mathbb{RP}^3\times\mathbb{CP}^1 \hookrightarrow \mathbb{RP}^\infty\times\mathbb{CP}^\infty$. *A bordism class* $[X^5, f]$ *equals* ±1 *if and only if* $\langle\alpha^3\cup\beta, f_*[X]\rangle = 1 \in \mathbb{Z}/2$. *There is a relation* $\langle \alpha\cup\beta^2, f_*[X]\rangle = 0$. *The action* τ_* *is the multiplication by* -1.

Proof. There is a product map

$$\varphi : \Omega_3^{\mathrm{Spin}}(\mathbb{RP}^\infty)\otimes\Omega_2^{\mathrm{Spin}}(\mathbb{CP}^\infty) \longrightarrow \Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty\times\mathbb{CP}^\infty),$$

induced by the product of manifolds. There is a corresponding product map on the Atiyah–Hirzebruch spectral sequences

$$\Phi : E^r_{p,q}(1)\otimes E^r_{s,t}(2) \longrightarrow E^r_{p+s,q+t}(3),$$

where on the E^∞-page Φ is compatible with the filtrations on the bordism groups and on the E^2-page it is just the cross-product map (see [23, p. 352]). It is easy to see that the Atiyah–Hirzebruch spectral sequence of $\Omega_2^{\mathrm{Spin}}(\mathbb{CP}^\infty)$ collapses on the line $p+q=2$. Also since $\Omega_3^{\mathrm{Spin}}(\mathbb{RP}^\infty)\cong\Omega_2^{\mathrm{Pin}^-}\cong\mathbb{Z}/8$ (by Kirby and Taylor [12]), we see that the Atiyah–Hirzebruch spectral sequence of $\Omega_3^{\mathrm{Spin}}(\mathbb{RP}^\infty)$ collapses on the line $p+q=3$. From the knowledge of $E^\infty_{5,0}(3)$ and $E^\infty_{4,1}(3)$ discussed above, we see that there are surjections

$$\Phi : E^\infty_{3,0}(1)\otimes E^\infty_{2,0}(2) \longrightarrow E^\infty_{5,0}(3), \quad \Phi : E^\infty_{2,1}(1)\otimes E^\infty_{2,0}(2) \longrightarrow E^\infty_{4,1}(3).$$

Therefore, φ is surjective. Now $\Omega_3^{\mathrm{Spin}}(\mathbb{RP}^\infty)\cong\mathbb{Z}/8$ is generated by $[\mathbb{RP}^3, \text{inclusion}]$ and $\Omega_2^{\mathrm{Spin}}(\mathbb{CP}^\infty)\cong\Omega_2^{\mathrm{Spin}}\oplus H_2(\mathbb{CP}^\infty)$. The group Ω_2^{Spin} is generated by T^2 with the Lie group spin structure. The product $\varphi(\mathbb{RP}^3, T^2)=0$, since the map $\mathbb{RP}^3\times T^2 \to \mathbb{RP}^\infty\times\mathbb{CP}^\infty$ factors through $\mathbb{RP}^\infty$

and $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty) = 0$. Therefore, we have a surjection

$$\mathbb{Z}/8 \otimes \mathbb{Z} \longrightarrow \Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty).$$

This shows that $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}/4$, generated by $\mathbb{R}\mathrm{P}^3 \times \mathbb{C}\mathrm{P}^1 \hookrightarrow \mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty$ and $[X, (\mathrm{id}_{\mathbb{R}\mathrm{P}^\infty} \times \tau) \circ f] = -[X, f]$. The fact that a bordism class $[X^5, f]$ equals ± 1 if and only if $\langle \alpha^3 \cup \beta, f_*[X] \rangle = 1 \in \mathbb{Z}/2$ comes from the fact that $E_{5,0}^\infty$ is dual to $\alpha^3\beta$. The relation $\langle \alpha \cup \beta^2, f_*[X] \rangle = 0$ comes from the fact that the dual of d_2 maps $\alpha\beta$ to $\alpha\beta^2$.

In general, on the E^∞-page of the Atiyah–Hirzebruch spectral sequence for $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r)$, the non-trivial terms in the line $p + q = 5$ are

$$\begin{aligned} E_{5,0}^\infty &= \bigoplus_i H_3(\mathbb{R}\mathrm{P}^\infty) \otimes H_2(\mathbb{C}\mathrm{P}_i^\infty) \oplus \bigoplus_{i \neq j} H_1(\mathbb{R}\mathrm{P}^\infty) \otimes H_2(\mathbb{C}\mathrm{P}_i^\infty) \otimes H_2(\mathbb{C}\mathrm{P}_j^\infty) \\ &\cong (\mathbb{Z}/2)^{r + r(r-1)/2}, \\ E_{4,1}^\infty &= \bigoplus_i H_2(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2) \otimes H_2(\mathbb{C}\mathrm{P}_i^\infty; \mathbb{Z}/2) \\ &\cong (\mathbb{Z}/2)^r. \end{aligned}$$

Using the same argument as in Lemma 4.3, we have the following proposition.

PROPOSITION 4.4 (Type II) *The bordism group $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r)$ is isomorphic to $(\mathbb{Z}/4)^r \oplus (\mathbb{Z}/2)^{r(r-1)/2}$. Let $\alpha \in H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2)$, $\beta_i \in H^2(\mathbb{C}\mathrm{P}_i^\infty; \mathbb{Z}/2)$ be the non-zero elements, and τ_i be the involution on $\mathbb{C}\mathrm{P}_i^\infty$ with $\tau_{i*} = -1$ on H_2. Then*

1. *the $\mathbb{Z}/2$-factors are determined by the invariants $\langle \alpha \cup \beta_i \cup \beta_j, f_*[X] \rangle \in \mathbb{Z}/2$, with $i, j = 1, \dots r$, and $i > j$;*
2. *a bordism class $[X, f]$ has component ± 1 in the ith $\mathbb{Z}/4$-factor if and only if $\langle \alpha^3 \cup \beta_i, f_*[X] \rangle = 1 \in \mathbb{Z}/2$, $i = 1, \dots r$;*
3. *there are relations $\langle \alpha \cup \beta_i^2, f_*[X] \rangle = 0$ for all i;*
4. *the action of τ_i on the bordism group is multiplication by -1 on the ith $\mathbb{Z}/4$-factor and trivial on other factors.*

Type III: the normal 2-type is

$$p : B = \mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r \times B\,\mathrm{Spin} \longrightarrow BO,$$

where the map on $\mathbb{R}\mathrm{P}^\infty$ is the classifying map of the vector bundle 2η. Therefore, the bordism group $\Omega_5(B, p)$ is the twisted Spin-bordism group

$$\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r; p_1^* 2\eta) = \tilde{\Omega}_7^{\mathrm{Spin}}(\mathrm{Th}(p_1^* 2\eta)).$$

In the Atiyah–Hirzebruch spectral sequence, the E^2-terms are

$$E_{p,q}^2 = \widetilde{H}_p(\mathrm{Th}(p_1^* 2\eta); \Omega_q^{\mathrm{Spin}}).$$

Since 2η is orientable, we may apply the Thom isomorphism and after a degree shift $p \mapsto p-2$, we have $E^2_{p,q} = H_p(\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r; \Omega_q^{\mathrm{Spin}})$. Therefore, the E^2-terms are the same as in the Type II case, and in the identification of the differentials d_2, we need to replace Sq^2 by $\mathrm{Sq}^2 + w_2(2\eta)$.

As before, we first look at the group $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; p_1^* 2\eta)$. Clearly, $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty; 2\eta) \cong \mathbb{Z}/16$ is a direct summand. Besides this, there are two terms on the E^∞-page at positions $(5, 0)$ and $(4, 1)$, respectively, each isomorphic to $\mathbb{Z}/2$. The extension problem is solved in the following lemma.

LEMMA 4.5 *We have an isomorphism*

$$\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; p_1^*(2\eta)) \cong \mathbb{Z}/4 \oplus \Omega_4^{\mathrm{Pin}^+}.$$

A bordism class $[X, f]$ has component ± 1 in the $\mathbb{Z}/4$-factor if and only if $\langle \alpha^3 \cup \beta, f_*[X] \rangle = 1 \in \mathbb{Z}/2$. There is a relation $\langle \alpha^3 \cup \beta, f_*[X] \rangle = \langle \alpha \cup \beta^2, f_*[X] \rangle$. The action τ_* of the involution τ on $\mathbb{C}\mathrm{P}^\infty$ is the multiplication by -1 on the $\mathbb{Z}/4$-factor and trivial on the $\Omega_4^{\mathrm{Pin}^+}$-factor.

Proof. From the above discussion we have

$$\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; p_1^*(2\eta)) \cong G \oplus \Omega_4^{\mathrm{Pin}^+},$$

where the order of G is 4. To determine G, the geometric argument in Lemma 4.3 does not work since now we have $\Omega_3^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty; 2\eta) \cong \Omega_2^{\mathrm{Pin}^+} = 0$. Thus, we turn to consider the Adams spectral sequence for $\widetilde{\Omega}^{\mathrm{Spin}}_{t-s-2}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; p_1^*(2\eta)) = \pi^S_{t-s}(\mathrm{Th}(p_1^*(2\eta)) \wedge M\mathrm{Spin})$ at prime 2:

$$\mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*(\mathrm{Th}(p_1^*(2\eta)) \wedge M\mathrm{Spin}; \mathbb{F}_2); \mathbb{F}_2) \Longrightarrow \pi^S_{t-s}(\mathrm{Th}(p_1^*(2\eta)) \wedge M\mathrm{Spin})/\text{non-2-torsion},$$

where $\mathcal{A}$ is the mod 2 Steenrod algebra. We have

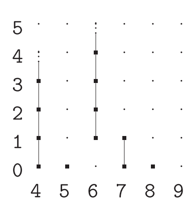
$$H^*(\mathrm{Th}(p_1^*(2\eta)) \wedge M\mathrm{Spin}; \mathbb{F}_2) \cong \widetilde{H}^*(\mathrm{Th}(p_1^*(2\eta)); \mathbb{F}_2) \otimes_{\mathbb{F}_2} H^*(M\mathrm{Spin}; \mathbb{F}_2),$$

where $\widetilde{H}^*(\mathrm{Th}(p_1^*(2\eta)); \mathbb{F}_2)$ is a free $\mathbb{F}_2[t, x]$-module on one generator u_2 of degree 2 (the Thom class), where $\deg t = 1$ and $\deg x = 2$, and

$$\mathrm{Sq}(u_2) = u_2 + t^2 u_2.$$

From this we may write down the $\mathcal{A}$-module structure of $H^*(\mathrm{Th}(p_1^*(2\eta)) \wedge M\mathrm{Spin}; \mathbb{F}_2)$ in degree ≤ 9, and produce a minimal free $\mathcal{A}$-resolution of $H^*(\mathrm{Th}(p_1^*(2\eta)) \wedge M\mathrm{Spin}; \mathbb{F}_2)$ which corresponds to the E_2-term of the spectral sequence. In practice, we may ignore the pure terms from $\mathbb{R}\mathrm{P}^\infty$, since we already know that the contribution of $\mathbb{R}\mathrm{P}^\infty$ is a $\mathbb{Z}/16$-summand.

In low degrees, the E_2-page of the spectral sequence is depicted as follows (with horizontal index $t-s$ and vertical index s. The calculation is confirmed by Olbermann and Abczynski using a computer program developed by Bruner):



This shows that $G \cong \mathbb{Z}/4$.

The fact that the generators of the $\mathbb{Z}/4$-factor are detected by the invariant $\langle \alpha^3 \cup \beta, f_*[X]\rangle \in \mathbb{Z}/2$ and the relation $\langle \alpha^3 \cup \beta, f_*[X]\rangle = \langle \alpha \cup \beta^2, f_*[X]\rangle$ are seen from the Atiyah–Hirzebruch spectral sequence, as in the Type II case. From this, we claim that $[X^5(0), f]$ represents a generator of $\mathbb{Z}/4$, where

$$f : X^5(0) \longrightarrow \mathbb{RP}^\infty \times \mathbb{CP}^\infty$$

is a normal 2-smoothing. To see this, recall that

$$X^5(0) = (\mathbb{RP}^5 - S^1 \times D^4) \cup_\partial (\mathbb{RP}^5 - S^1 \times D^4),$$

and $\mathbb{RP}^5 - S^1 \times D^4$ is the disk bundle $D(2\eta)$ over $\mathbb{RP}^3$. Therefore, $X^5(0)$ is actually the sphere bundle $S(2\eta \oplus \mathbb{R})$. The cohomology groups are easily computed and we see that $\langle \alpha^3 \cup \beta, f_*[X]\rangle = 1$.

Now let $r : X^5(0) \to X^5(0)$ be the fiberwise antipodal map; we have a commutative diagram

$$\begin{tikzcd} X^5(0) \arrow[r, "f"] \arrow[d, "r"] & \mathbb{RP}^\infty \times \mathbb{CP}^\infty \arrow[d, "(\mathrm{id}, \tau)"] \\ X^5(0) \arrow[r, "f"] & \mathbb{RP}^\infty \times \mathbb{CP}^\infty \end{tikzcd}$$

Since r is orientation reversing, we conclude that the action of τ on the $\mathbb{Z}/4$ factor is multiplication by -1. It is also clear that the action of τ on the $\Omega_4^{\mathrm{Pin}^+}$ is trivial.

In the general situation, the calculation is similar, and we have the following proposition.

PROPOSITION 4.6 (Type III) *The bordism group* $\Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty \times (\mathbb{CP}^\infty)^r; p_1^*(2\eta))$ *is isomorphic to* $(\mathbb{Z}/4)^r \oplus (\mathbb{Z}/2)^{r(r-1)/2} \oplus \Omega_4^{\mathrm{Pin}^+}$. *Furthermore,*

(1) *the* $\mathbb{Z}/2$*-factors are determined by the invariants* $\langle \alpha \cup \beta_i \cup \beta_j, f_*[X]\rangle \in \mathbb{Z}/2$, *with* $i, j = 1, \dots r$, *and* $i > j$;
(2) *a bordism class* $[X, f]$ *has component* ± 1 *in the* i*th* $\mathbb{Z}/4$*-factor if and only if* $\langle \alpha^3 \cup \beta_i, f_*[X]\rangle = 1 \in \mathbb{Z}/2$, $i = 1, \dots r$;
(3) *there are relations* $\langle \alpha \cup \beta_i^2, f_*[X]\rangle = \langle \alpha^3 \cup \beta_i, f_*[X]\rangle$ *for all* i;
(4) *the action* τ_i *on the bordism group is the multiplication by* -1 *on the* i*th* $\mathbb{Z}/4$*-factor and trivial on other factors.*

Type I: recall that the normal 2-type is

$$p : B = \mathbb{RP}^\infty \times (\mathbb{CP}^\infty)^r \times B\,\mathrm{Spin} \longrightarrow BO,$$

where the map p on the first $\mathbb{CP}^\infty$ is the classifying map of the vector bundle γ. Therefore, the bordism group $\Omega_5(B, p)$ is the twisted Spin-bordism group

$$\Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty \times (\mathbb{CP}^\infty)^r; p_2^*\gamma) = \tilde{\Omega}_7^{\mathrm{Spin}}(\mathrm{Th}(p_2^*\gamma)).$$

As before, we apply the Thom isomorphism and the E^2-terms in the Atiyah–Hirzebruch spectral sequence are $E^2_{p,q} = \widetilde{H}_p(\mathbb{RP}^\infty \times (\mathbb{CP}^\infty)^r; \Omega_q^{\mathrm{Spin}})$, where in the identification of the differentials d_2, we replace Sq^2 by $\mathrm{Sq}^2 + w_2(\gamma)$. The calculation is analogous to the Type II case.

PROPOSITION 4.7 (Type I) *The bordism group* $\Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty \times (\mathbb{CP}^\infty)^r; p_2^*\gamma)$ *is isomorphic to* $(\mathbb{Z}/4)^{r-1} \oplus (\mathbb{Z}/2)^{r(r-1)/2} \oplus \Omega_4^{\mathrm{Pin}^c}$. *Furthermore,*

(1) *the* $\mathbb{Z}/2$*-factors are determined by the invariants* $\langle \alpha \cup \beta_i \cup \beta_j, f_*[X]\rangle \in \mathbb{Z}/2$, *with* $i, j = 1, \dots r$, *and* $i > j$;
(2) *a bordism class* $[X, f]$ *has component* ± 1 *in the* i*th* $\mathbb{Z}/4$*-factor if and only if* $\langle \alpha^3 \cup \beta_i f_*[X]\rangle = 1 \in \mathbb{Z}/2$, $i = 2, \dots, r$;
(3) *there are relations* $\langle \alpha^5 + \alpha^3 \cup \beta_1, f_*[X]\rangle = 0$ *and* $\langle \alpha \cup \beta_i^2, f_*[X]\rangle = \langle \alpha \cup \beta_1 \cup \beta_i, f_*[X]\rangle$ *for all* i;
(4) *the action* τ_i $(i \geq 2)$ *on the bordism group is the multiplication by* -1 *on the* i*th* $\mathbb{Z}/4$*-factor and trivial on other factors.*

5. Proofs of the main results

In this section, we prove Theorems 3.1 and 3.6. From the point of view of Proposition 4.1, the key point in Theorem 3.1 is to show that for manifolds having the same invariants stated in the theorem, we can find appropriate normal 2-smoothings in B, such that they are bordant in $\Omega_5(B, p)$. In some applications, this is done by understanding the action of the group of fiber homotopy equivalences $\mathrm{Aut}(B, p)$ on $\Omega_n(B, p)$. However, in our situation we find it more practical to produce the smoothings directly.

LEMMA 5.1 *Let* M^5 *be a fibered-type manifold with* $\pi_1(M) \cong \mathbb{Z}/2$ *and* $H_2(M) \cong \mathbb{Z}^r$. *Let* $t \in H^1(M; \mathbb{Z}/2)$ *be the non-zero element, and let* $\{t^2, x_1, \dots, x_r\}$ *be a basis of* $H^2(M; \mathbb{Z}/2)$. *Then* $\{t^3, tx_1, \dots, tx_r\}$ *is a basis of* $H^3(M; \mathbb{Z}/2)$.

Proof. Consider the Leray–Serre cohomology spectral sequence for the fibration $\tilde{M} \to M \to \mathbb{RP}^\infty$ with $\mathbb{Z}/2$-coefficients. Note that $\dim H^2(M; \mathbb{Z}/2) = r + 1$ and $\dim H^2(\tilde{M}; \mathbb{Z}/2) = r$. This implies that the differential

$$d_2 : E_2^{0,2} = H^2(\tilde{M}; \mathbb{Z}/2) \to E_3^{3,0} = H^3(\mathbb{RP}^\infty; \mathbb{Z}/2)$$

must be trivial. Therefore, the elements $t^3, tx_1, \dots, tx_r$ all survive to form a basis of $H^3(M; \mathbb{Z}/2)$.

Proof of Theorem 3.1 First of all, by Lemma 2.6, we see that in the Types II and III cases, $[P] \in \Omega_4^{\mathrm{Pin}^\pm}/\pm$ is an invariant for M. Since we do not have a statement for Pin^c, we shall give an alternative argument below for the Type I case.

Let $f : M^5 \to \mathbb{RP}^\infty$ be the classifying map of π_1, $t = f^*\alpha \in H^1(M; \mathbb{Z}/2)$. Consider the non-degenerate symmetric bilinear form

$$\lambda : H^2(M; \mathbb{Z}/2) \times H^2(M; \mathbb{Z}/2) \xrightarrow{\cup} H^4(M; \mathbb{Z}/2) \xrightarrow{\cup t} H^5(M; \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

Type II: note that since $\langle t^5, [M]\rangle = \langle \alpha^5, f_*[M]\rangle$ and $\Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty) = 0$, we have $\lambda(t^2, t^2) = 0$. From this and the relations in Proposition 4.4, we see that $\lambda(x, x) = 0$ for all x. Therefore, we may extend t^2 to a symplectic basis of λ, $\{t^2, u_1, \dots, u_r\}$. Especially, we have $\lambda(t^2, u_1) = 1, \lambda(u_1, u_1) = 0$ and $\lambda(t^2, u_i) = \lambda(u_1, u_i) = 0$ for $i > 1$. Now let $u_i' = u_i + u_1$ for $i > 1$; then $\lambda(t^2, u_i') = 1$ for all

i and $\lambda(u'_i, u'_j) = \lambda(u_i, u_j)$. We may lift $\{u'_1, \ldots, u'_r\}$ to a basis of the free part of $H^2(M)$ and get a map $M \to (\mathbb{C}\mathrm{P}^\infty)^r$. Together with the canonical map $f : M \to \mathbb{R}\mathrm{P}^\infty$ and the classifying map of a Spin-structure $M \to B\,\mathrm{Spin}$, we obtain a normal 2-smoothing $\bar{\nu} : M \to B = \mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r \times B\,\mathrm{Spin}$.

Now suppose that M' is another manifold, with a normal 2-smoothing $\bar{\nu}'$ constructed as above. Then, by Proposition 4.4 (composing $\bar{\nu}'$ with some τ_i to interchange ± 1 in the $\mathbb{Z}/4$-factors if necessary) $[M, \bar{\nu}] = [M', \bar{\nu}'] \in \Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r)$. Proposition 4.1 implies that they are diffeomorphic.

For the other two cases, the procedure of finding an appropriate map to $(\mathbb{C}\mathrm{P}^\infty)^r$ is similar, and thus we omit the details.

Type III: first note that by the relation in Proposition 4.6, for all $x \in H^2(M; \mathbb{Z}/2)$, $\lambda(t^2, x) = \lambda(x, x)$. There are two different cases:

(1) if $\lambda(t^2, t^2) = 0$: then there exists a u_1 such that $\lambda(t^2, u_1) = 1$. On the orthogonal complement of $\mathrm{span}(t^2, u_1)$, we have $\lambda(x, x) = 0$, thus there exists a symplectic basis $\{u_2, \ldots, u_r\}$. Then the argument is the same as in the previous case.
(2) if $\lambda(t^2, t^2) = 1$: let U be the orthogonal complement of $\mathrm{span}(t^2)$; then $\lambda(x, x) = \lambda(t^2, x) = 0$ for all $x \in U$. There exists a symplectic basis of U, $\{u_1, \ldots, u_r\}$. Let $u'_i = u_i + t^2$; then $\lambda(t^2, u'_i) = 1$ for all i and $\lambda(u'_i, u'_j) = \lambda(u_i, u_j) + 1$. The remaining argument is the same as in the previous case.

For M and M' having the same rank $H_2 = r$, like in the Type II case, we may use maps to $(\mathbb{C}\mathrm{P}^\infty)^r$ constructed above to make the corresponding bordism classes have equal $(\mathbb{Z}/4)^r \oplus (\mathbb{Z}/2)^{r(r-1)/2}$-component. Now if M and M' have $[P] = [P'] \in \Omega_4^{\mathrm{Pin}^+}/\pm$, then, by choosing an appropriate Spin-structure on $TM \oplus f^*(2\eta)$, we may make the $\Omega_4^{\mathrm{Pin}^+}$-component equal.

Type I: let $u_1 = w_2(M)$. By the relation in Proposition 4.7, for all $x \in H^2(M; \mathbb{Z}/2)$, $\lambda(u_1, x) = \lambda(x, x)$. To find the map to $(\mathbb{C}\mathrm{P}^\infty)^r$, we have four cases:

(1) if $\lambda(t^2, t^2) = 1$ and $\lambda(u_1, u_1) = 0$: then λ is non-degenerate on $\mathrm{span}(t^2, u_1)$. Let U be the orthogonal complement of $\mathrm{span}(t^2, u_1)$; then, for all $x \in U$, $\lambda(x, x) = 0$. There exists a symplectic basis $\{u_2, \ldots, u_r\}$.
(2) if $\lambda(t^2, t^2) = 0$ and $\lambda(u_1, u_1) = 0$: then exists a u_2 such that $\lambda(u_1, u_2) = 1$ and $\lambda(t^2, u_2) = 0$. We see that λ is non-degenerate on $\mathrm{span}(u_1, u_2)$. On the orthogonal complement we have $\lambda(x, x) = 0$. Therefore, there is a symplectic basis $\{t^2, u_3, \ldots, u_r\}$.
(3) if $\lambda(t^2, t^2) = 1$ and $\lambda(u_1, u_1) = 1$: let U be the orthogonal complement of $\mathrm{span}(u_1)$, then there exists a symplectic basis $\{u_2, u_3, \ldots, u_r\}$ for U and we may choose $u_2 = t^2 + u_1$.
(4) if $\lambda(t^2, t^2) = 0$ and $\lambda(u_1, u_1) = 1$: then, on the orthogonal complement of $\mathrm{span}(x_1)$, there is a symplectic basis $\{t^2, u_2, \ldots, u_r\}$.

Now we need to consider the $\Omega_4^{\mathrm{Pin}^c}$-component. Note that since the manifolds in the list given in Theorem 3.6 exhaust all possible values of $\mathrm{rank}\, H_2$ and $[P]$, an M of Type I must be diffeomorphic to some manifolds in the list. Now we just need to show that the manifolds in the list are not diffeomorphic to each other.

The s-component of $[P] \in \Omega_4^{\mathrm{Pin}^c}$ is determined by $w_2(P)^2$, therefore varying Pin^c-structures on P^4 will not change the s-component. Thus, we see that the two subfamilies

$$X^5(q) \sharp_{S^1} (S^2 \times \mathbb{R}\mathrm{P}^3) \sharp_{S^1} ((\sharp_k S^2 \times S^2) \times S^1)$$

and

$$X^5(q)\sharp_{S^1}(\mathbb{C}P^2 \times S^1)\sharp_{S^1}((\sharp_k S^2 \times S^2) \times S^1)$$

do not have coincidence.

Let Q^4 be a characteristic submanifold of $X^5(q)$; then a characteristic submanifold of $X^5(q)\sharp_{S^1}(S^2 \times \mathbb{R}P^3)\sharp_{S^1}((\sharp_k S^2 \times S^2) \times S^1)$ can be taken as $P = Q\sharp(S^2 \times \mathbb{R}P^2)\sharp(S^2 \times S^2)$. We have $[S^2 \times \mathbb{R}P^2] = [S^2 \times S^2] = 0 \in \Omega_4^{\mathrm{Pin}^c}$. So we see $[P] = q \in \Omega_4^{\mathrm{Pin}^c}/\pm$ and different q's give non-diffeomorphic $X^5(q)\sharp_{S^1}(S^2 \times \mathbb{R}P^3)\sharp_{S^1}((\sharp_k S^2 \times S^2) \times S^1)$. Similarly for the manifolds $X^5(q)\sharp_{S^1}(\mathbb{C}P^2 \times S^1)\sharp_{S^1}((\sharp_k S^2 \times S^2) \times S^1)$, since $[\mathbb{C}P^2] = (0, 1) \in \Omega_4^{\mathrm{Pin}^c}$.

The relations among the invariants are essentially seen in the previous proof, but there is a more conceptual way to see this.

Proof of Theorem 3.6 We use the semi-characteristic class defined by Lee [**17**]. We work with $\mathbb{Q}$-coefficient, in this case, the semi-characteristic class of an odd-dimensional manifold with a free $\mathbb{Z}/2$-action is a homomorphism

$$\chi_{1/2} : \Omega_5(\mathbb{Z}/2) \longrightarrow L^5(\mathbb{Q}[\mathbb{Z}/2]) \cong \mathbb{Z}/2,$$

where $\Omega_5(\mathbb{Z}/2)$ is the bordism group of closed smooth oriented manifolds with an orientation-preserving free $\mathbb{Z}/2$-action, and $L^5(\mathbb{Q}[\mathbb{Z}/2])$ is the symmetric L-group of the rational group ring $\mathbb{Q}[\mathbb{Z}/2]$. We refer to [**4**, **17**] for details.

Let M^5 be an oriented smooth 5-manifold with fundamental group $\mathbb{Z}/2$; then the semi-characteristic class $\chi_{1/2}(\tilde{M}; \mathbb{Q}) \in \mathbb{Z}/2$ is defined. There is a characteristic class formula [**4**, Theorem C]

$$\chi_{1/2}(\tilde{M}; \mathbb{Q}) = \langle w_4(M) \cup f^*(\alpha), [M]\rangle,$$

where $f : M \to \mathbb{R}P^\infty$ is the classifying map of the covering and $\alpha \in H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)$ is the non-zero element. On the other hand, $\chi_{1/2}(\tilde{M}; \mathbb{Q})$ is identified with (see [**4**, p. 57])

$$\begin{aligned}\hat{\chi}_{1/2}(\tilde{M}; \mathbb{Q}) &:= \dim_{\mathbb{Q}} H_0(\tilde{M}; \mathbb{Q}) + \dim_{\mathbb{Q}} H_1(\tilde{M}; \mathbb{Q}) + \dim_{\mathbb{Q}} H_2(\tilde{M}; \mathbb{Q}) \pmod 2 \\ &\equiv 1 + r \pmod 2.\end{aligned}$$

Type II: the Wu classes of M are $v_1 = 0$ and $v_2 = 0$ since $w_1(M) = w_2(M) = 0$. Therefore, $w_4(M) = \mathrm{Sq}^2 v_2 = 0$. This means that r is odd.

Type III: the Wu classes of M are $v_1 = 0$ and $v_2 = w_2(M) = t^2$. Therefore, $w_4(M) = \mathrm{Sq}^2 v_2 = t^4$ and $\langle w_4(M) \cup f^*(\alpha), [M]\rangle = \langle \alpha^5, \bar{\nu}_*[M]\rangle$. By the Atiyah–Hirzebruch spectral sequence, there is a non-split exact sequence

$$0 \longrightarrow \mathbb{Z}/8 \longrightarrow \Omega_5^{\mathrm{Spin}}(\mathbb{R}P^\infty; 2\eta) \longrightarrow H_5(\mathbb{R}P^\infty) \longrightarrow 0.$$

Note that the bordism class $[M, \bar{\nu}] \in \Omega_5^{\mathrm{Spin}}(\mathbb{R}P^\infty; 2\eta)$ corresponds to the Pin^+-bordism class of a characteristic submanifold, which we denote by q. Therefore, $\bar{\nu}_*[M] \equiv q \pmod 2$. This implies that $r + q$ is odd.

Type I: the Wu classes of M are $v_1 = 0$ and $v_2 = w_2(M) = \bar{\nu}^* w_2(\gamma)$. Therefore, $w_4(M) = \mathrm{Sq}^2 v_2 = \bar{\nu}^* w_2(\gamma)^2$ and $\langle w_4(M) \cup f^*(\alpha), [M] \rangle = \langle \alpha \cup \beta^2, \bar{\nu}_*[M] \rangle$. Check on the generators of $\Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty \times \mathbb{CP}^\infty; \gamma)$, $\mathbb{RP}^5 \sharp_{S^1} (S^2 \times \mathbb{RP}^3)$ with $(q = 1, s = 0)$ and $\mathbb{RP}^5 \sharp_{S^1} (\mathbb{CP}^2 \times S^1)$ with $(q = 1, s = 1)$; it is seen that $\langle \alpha \cup \beta^2, \bar{\nu}_*[M] \rangle \equiv q + s \pmod 2$. This implies the relation $q + s + r \equiv 1 \pmod 2$.

6. Circle bundles over 1-connected 4-manifolds

As an application of the main results, in this section we study the classification of certain circle bundles over simply connected 4-manifolds.

Let X^4 be a simply connected 4-manifold, smooth or topological, and let ξ be a complex line bundle over X, with first Chern class $c_1(\xi) \in H^2(X; \mathbb{Z})$. Choose a Riemannian metric on ξ, and then the total space of the corresponding circle bundle is a 5-manifold M. The homotopy long exact sequence of the fiber bundle shows that $\pi_1(M) \cong \mathbb{Z}/m$ if $c_1(\xi)$ is an m-multiple of a primitive element.

In [5], a classification of M in terms of the topological invariants of X and $c_1(\xi)$ is obtained for $m = 1$, using the classification theorem of Smale and Barden. It is also known that $H_2(M)$ is torsion-free of rank $H_2(X) - 1$ and that M is of fibered type. In this section, we apply the classification results to the $m = 2$ case, to give the classification of M in terms of the topological invariants of X and $c_1(\xi)$. We also identify M in the list of standard forms in Theorems 3.7 and 3.11.

6.1. *Invariants of M*

In this subsection, we collect the basic algebraic-topological invariants of M.

PROPOSITION 6.1 *Let M^5 be a circle bundle over a simply connected 4-manifold X, with first Chern class $c_1(\xi) = 2 \cdot$ (primitive); then*

(1) $\pi_1(M) \cong \mathbb{Z}/2$;
(2) $H_2(M) \cong \mathbb{Z}^r$ *where* $r = \operatorname{rank} H_2(X) - 1$;
(3) *the* $\pi_1(M)$*–action on* $H_2(\widetilde{M})$ *is trivial*;
(4) *the type of* M^5 *is given by*

Type I	*Type* II	*Type* III
$w_2(X) \neq 0$ and $w_2(X) \not\equiv c_1(\widetilde{\xi}) \pmod 2$	$w_2(X) = 0$	$w_2(X) \equiv c_1(\widetilde{\xi}) \pmod 2$

Proof. First of all, the homotopy long exact sequence

$$\pi_1(S^1) \longrightarrow \pi_1(M) \longrightarrow \pi_1(X)$$

implies that $\pi_1(M)$ is a cyclic group. The Gysin sequence

$$0 \longrightarrow H_2(M) \longrightarrow H_2(X) \xrightarrow{\cap c_1} H_0(X) \longrightarrow H_1(M) \longrightarrow 0$$

shows that $H_2(M)$ is torsion-free of rank equal to rank $H_2(X) - 1$ and $H_1(M) \cong \mathbb{Z}/2$ since $c_1(\xi) = 2 \cdot$ (primitive). Note that the universal cover $\tilde{M}$ is a circle bundle over X, denoted by $\tilde{\xi}$, with first Chern class $c_1(\tilde{\xi}) = \frac{1}{2}c_1(\xi)$. The $\pi_1(M)$-action on $\tilde{M}$ is the antipodal map on each fiber, and thus the commutative diagram

$$\begin{tikzcd} H_2(\tilde{M}) \arrow[r, "T_*"] \arrow[dr, "p_*"'] & H_2(\tilde{M}) \arrow[d, "p_*"] \\ & H_2(X) \end{tikzcd}$$

shows that the action on $H_2(\tilde{M})$ is trivial. For the Stiefel–Whitney class, if X is smooth, we have $TM \oplus \mathbb{R} = p^*(TX \oplus \xi)$ (where p is the projection map); this implies $w_2(M) = p^*w_2(X)$. In general, $X - pt$ admits a smooth structure; then the same argument holds; see [**5**, Lemma 3].

6.2. *Smoothings of M*

Proposition 6.2 *Let $\xi : S^1 \hookrightarrow M^5 \to X$ be a non-trivial circle bundle over a closed, simply connected, topological 4-manifold. If $c_1(\xi)$ is an odd multiple of a primitive element, then M is smoothable; if $c_1(\xi)$ is an even multiple of a primitive element, then M admits a smooth structure if and only if* $\mathrm{KS}(X) = 0$.

Proof. Let M^5 be a topological 5-manifold; then, by Kirby and Siebenmann [**11**], the obstruction for smoothing M lies in $H^4(M; \pi_3(\mathrm{Top}/O)) = H^4(M; \pi_3(\mathrm{Top}/PL)) = H^4(M; \mathbb{Z}/2) \cong H_1(M; \mathbb{Z}/2)$. The latter group is trivial if $c_1(\xi)$ is an odd multiple of a primitive element. On the other hand, we have $TM \oplus \mathbb{R} = \pi^*(TX \oplus \xi)$, where π is the projection map. Therefore, the obstruction for smoothing M is $\pi^*\mathrm{KS}(X)$. It is seen from the Gysin sequence that $\pi^* : H^4(X; \mathbb{Z}/2) \to H^4(M; \mathbb{Z}/2)$ is injective if $c_1(\xi)$ is an even multiple of a primitive element. Therefore, M admits a smooth structure if and only if $\mathrm{KS}(X) = 0$.

Now we give a geometric description of the characteristic submanifold of a circle bundle over simply connected X^4.

Lemma 6.3 *Let $\xi : S^1 \hookrightarrow M^5 \to X$ be a circle bundle, $\pi_1(M) \cong \mathbb{Z}/2$. Let $F \subset X$ be an embedded surface dual to $c_1(\tilde{\xi})$, $N(F)$ be a tubular neighborhood of F in X and $S^1 \hookrightarrow B \to F$ be the restriction of ξ on F. Then there is a double cover map $\partial N(F) \to B$ and the characteristic submanifold of M is $P^4 = (X - \mathring{N}(F)) \cup_\partial B$.*

In other words, the characteristic submanifold P is obtained by removing a tubular neighborhood of an embedded surface dual to $c_1(\tilde{\xi})$ and then identifying antipodal points on each fiber.

Proof. Since $c_1(\xi) = 2\cdot$(primitive), the circle bundle is the pullback of the circle bundle over $\mathbb{C}\mathrm{P}^2$ with first Chern class $= 2\cdot$(primitive):

$$\begin{tikzcd} S^1 \arrow[r, "="] \arrow[d] & S^1 \arrow[d] & \\ M^5 \arrow[r, "f"] \arrow[d] & \mathbb{R}\mathrm{P}^5 \arrow[d] & \supset \mathbb{R}\mathrm{P}^4 = \mathbb{R}\mathrm{P}^3 \cup D^4 \\ X \arrow[r, "g"] & \mathbb{C}\mathrm{P}^2 & = \mathbb{C}\mathrm{P}^1 \cup D^4 \end{tikzcd}$$

Now $P = f^{-1}(\mathbb{R}\mathrm{P}^4) = f^{-1}(D^4 \cup_{S^3} \mathbb{R}\mathrm{P}^3)$. Let $F = g^{-1}(\mathbb{C}\mathrm{P}^1)$ be the transverse preimage of $\mathbb{C}\mathrm{P}^1$; then the normal bundle ν of F in X is the pullback of the Hopf bundle, and the restriction of ξ on F is $\nu \otimes \nu$, therefore there is a double cover $\partial N(F) \to B$. It is easy to see that $P^4 = (X - \mathring{N}(F)) \cup_\partial B$.

LEMMA 6.4 *Let P be as above. Then* $\mathrm{KS}(P) = \mathrm{KS}(X)$.

Proof. We identify $N(F)$ with the normal 2-disk bundle. Let V be the associated $\mathbb{R}\mathrm{P}^2$-bundle obtained by identifying antipodal points on $\partial N(F)$. Then, by the construction,

$$P = X \cup_{N(F)\times\{0\}} N(F) \times I \cup_{N(F)\times\{1\}} V.$$

Therefore, P is bordant to $X \sqcup V$. It was shown by Hsu [10] and Lashof–Taylor [15] that the Kirby–Siebenmann invariant is a bordism invariant, thus $\mathrm{KS}(P) = \mathrm{KS}(X) + \mathrm{KS}(V) = \mathrm{KS}(X)$ since V is smooth.

6.3. *Classification*

Now we can give a classification of circle bundles over 1-connected 4-manifolds, and identify them with the standard forms in Theorems 3.7 and 3.11, in terms of the topology of X and ξ.

For the Type II manifolds, the classification is an immediate consequence of Theorems 3.1 and 3.4.

THEOREM 6.5 (Type II) *Let X be a closed, simply connected, topological spin 4-manifold, and $\xi : S^1 \hookrightarrow M^5 \to X$ be a circle bundle over X with $c_1(\xi) = 2\cdot$ (primitive). Then we have the following conditions*:

(1) *if* $\mathrm{KS}(X) = 0$, *then M is smoothable and M is diffeomorphic to*

$$(S^2 \times \mathbb{R}\mathrm{P}^3)\sharp_{S^1}((\sharp_k S^2 \times S^2) \times S^1),$$

(2) *if* $\mathrm{KS}(X) = 1$, *then M is non-smoothable and M is homeomorphic to*

$$*(S^2 \times \mathbb{R}\mathrm{P}^3)\sharp_{S^1}((\sharp_k S^2 \times S^2) \times S^1),$$

where $k = \mathrm{rank}\, H_2(X)/2 - 1$.

REMARK 6.6 Note that, for a spin 4-manifold X, rank $H_2(X)$ is even and thus k is an integer.

For smooth manifolds of Type III, we do not know a good invariant detecting the bordism group $\Omega_4^{\mathrm{Pin}^+}$. Therefore, we could only determine the diffeomorphism type up to an ambiguity of order 2. This is based on the following exact sequence (see [12, Section 5])

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \Omega_4^{\mathrm{Pin}^+} \xrightarrow{\bigcap w_1^2} \Omega_2^{\mathrm{Pin}^-} \longrightarrow 0,$$

where $\bigcap w_1^2$ is the operation of taking a submanifold dual to w_1^2. The generators of $\Omega_2^{\mathrm{Pin}^-}$ are $\pm\mathbb{R}\mathrm{P}^2$ and $\bigcap w_1^2$ maps $\pm\mathbb{R}\mathrm{P}^4$ to $\pm\mathbb{R}\mathrm{P}^2$. The image of $[P]$ in $\Omega_2^{\mathrm{Pin}^-}$ can be determined from the data of the circle bundle.

In the topological case, we have an epimorphism (see [12, Section 9])

$$\Omega_4^{\mathrm{TopPin}^+} \longrightarrow \Omega_2^{\mathrm{TopPin}^-} \cong \mathbb{Z}/8,$$

which is an isomorphism on the subgroup generated by $\mathbb{R}\mathrm{P}^4$. By Lemma 6.4 we have $\mathrm{KS}(P) = \mathrm{KS}(X)$. Therefore, by Theorem 3.4, we have a complete topological classification.

THEOREM 6.7 (Type III) *Let X be a closed, simply connected topological 4-manifold, and let $\xi : S^1 \hookrightarrow M^5 \to X$ be a circle bundle over X with $c_1(\xi) = 2\cdot$(primitive) and $w_2(X) \equiv c_1(\tilde{\xi}) \pmod 2$. Then we have the following conditions*:

(1) *if X is smooth, then the diffeomorphism type of M (with the induced smooth structure) is determined up to an ambiguity of order 2 by* rank $H_2(X)$ *and* $\langle c_1(\tilde{\xi})^2, [X]\rangle \in (\mathbb{Z}/8)/\pm = \{0, 1, 2, 3, 4\}$.
(2) *M is homeomorphic to $X^5(p,q)\sharp_{S^1}((\sharp_k S^2 \times S^2) \times S^1)$, where $q = \langle c_1(\tilde{\xi})^2, [X]\rangle \in (\mathbb{Z}/8)/\pm = \{0, 1, 2, 3, 4\}$, $k = (\mathrm{rank}\, H_2(X) - (3 + (-1)^q)/2)/2$, and $p = \mathrm{KS}(X)$.*

Proof. We only need to prove (1), since the proof of (2) is similar. We see from the proof of Lemma 6.3 that $P = f^{-1}(\mathbb{R}\mathrm{P}^4)$, where $f : P \to \mathbb{R}\mathrm{P}^4$ induces an isomorphism on π_1. If the mod 2 degree of f is 1, then the submanifold dual to $w_1(P)$ is $f^{-1}(\mathbb{R}\mathrm{P}^3)$, and the submanifold V dual to $w_1(P)^2$ is $f^{-1}(\mathbb{R}\mathrm{P}^2)$. Now we have the following commutative diagram:

$$\begin{array}{ccccc} S^1 & \xrightarrow{=} & S^1 & & \\ \downarrow & & \downarrow & & \\ \partial N(F) & \xrightarrow{f} & \mathbb{R}\mathrm{P}^3 & & \supset \mathbb{R}\mathrm{P}^2 = D^2 \cup S^1 \\ \downarrow & & \downarrow & & \\ F & \xrightarrow{g} & \mathbb{C}\mathrm{P}^1 & & = D^2 \cup pt \end{array}$$

Let $d = \deg g = \langle c_1(\tilde{\xi})^2, [X]\rangle$ and $D = g^{-1}(pt) = \{p_1, \dots, p_d\}$. It is seen that $V = f^{-1}(\mathbb{R}\mathrm{P}^2) = (F - D) \cup_\partial d \cdot S^1$ (where the glueing map is of degree 2) and $[V] = d \cdot [\mathbb{R}\mathrm{P}^2] \in \Omega_2^{\mathrm{Pin}^-}$. If the mod 2 degree of f is zero, then we consider the circle bundle over $X\sharp\mathbb{C}\mathrm{P}^2$ with first Chern class $(c_1(\xi), 2)$.

The corresponding map has non-zero mod 2 degree; the image of the corresponding characteristic submanifold in $\Omega_2^{\mathrm{Pin}^-}$ equals that of the original one plus 1. Finally, $\langle (c_1(\tilde{\xi}), 1)^2, [X \sharp \mathbb{C}\mathrm{P}^2] \rangle = \langle c_1(\tilde{\xi})^2, [X] \rangle + 1$. This proves the theorem.

For the manifolds of Type I, we have the following theorem.

THEOREM 6.8 (Type I) *Let X be a closed, simply connected non-spin topological 4-manifold, and $\xi : S^1 \hookrightarrow M^5 \to X$ be a circle bundle over X with $c_1(\xi) = 2\cdot$ (primitive) and $w_2(X) \not\equiv c_1(\tilde{\xi}) \pmod 2$. We have the following conditions*:

(1) *if* $\mathrm{KS}(X) = 0$, *then M is smoothable and*
(i) *if* $\langle w_2(X)^2, [X] \rangle \equiv \langle c_1(\tilde{\xi})^2, [X] \rangle \pmod 2$, *then M is diffeomorphic to*

$$X^5(q) \sharp_{S^1} (S^2 \times \mathbb{R}\mathrm{P}^3) \sharp_{S^1} ((\sharp_k S^2 \times S^2) \times S^1),$$

where $q = \langle c_1(\tilde{\xi})^2, [X] \rangle \in (\mathbb{Z}/8)/\pm = \{0, 1, 2, 3, 4\}$ *and*

$$k = \tfrac{1}{2}(\mathrm{rank}\, H_2(X) - \tfrac{1}{2}(7 + (-1)^q));$$

(ii) *if* $\langle w_2(X)^2, [X] \rangle \not\equiv \langle c_1(\tilde{\xi})^2, [X] \rangle \pmod 2$, *then M is diffeomorphic to*

$$X^5(q) \sharp_{S^1} (\mathbb{C}\mathrm{P}^2 \times S^1) \sharp_{S^1} ((\sharp_k S^2 \times S^2) \times S^1),$$

where $q = \langle c_1(\tilde{\xi})^2, [X] \rangle \in (\mathbb{Z}/8)/\pm = \{0, 1, 2, 3, 4\}$ *and*

$$k = \tfrac{1}{2}(\mathrm{rank}\, H_2(X) - \tfrac{1}{2}(5 + (-1)^q)),$$

(2) *if* $\mathrm{KS}(X) = 1$, *then M is non-smoothable and*
(i) *if* $\langle w_2(X)^2, [X] \rangle \equiv \langle c_1(\tilde{\xi})^2, [X] \rangle \pmod 2$, *then M is homeomorphic to*

$$X^5(1, q) \sharp_{S^1} (S^2 \times \mathbb{R}\mathrm{P}^3) \sharp_{S^1} ((\sharp_k\, S^2 \times S^2) \times S^1),$$

where $q = \langle c_1(\tilde{\xi})^2, [X] \rangle \in (\mathbb{Z}/8)/\pm = \{0, 1, 2, 3, 4\}$ *and*

$$k = \tfrac{1}{2}(\mathrm{rank}\, H_2(X) - \tfrac{1}{2}(7 + (-1)^q));$$

(ii) *if* $\langle w_2(X)^2, [X] \rangle \not\equiv \langle c_1(\tilde{\xi})^2, [X] \rangle \pmod 2$, *then M is homeomorphic to*

$$X^5(1, q) \sharp_{S^1} (\mathbb{C}\mathrm{P}^2 \times S^1) \sharp_{S^1} ((\sharp_k S^2 \times S^2) \times S^1),$$

where $q = \langle c_1(\tilde{\xi})^2, [X] \rangle \in (\mathbb{Z}/8)/\pm = \{0, 1, 2, 3, 4\}$ *and*

$$k = \tfrac{1}{2}(\mathrm{rank}\, H_2(X) - \tfrac{1}{2}(5 + (-1)^q)).$$

REMARK 6.9 Note that, for a 4-manifold X, $\langle w_2(X)^2, [X] \rangle \equiv \mathrm{rank}\, H_2(X) \pmod 2$. This ensures that k is an integer.

Proof. We only need to prove (1); the proof of (2) is similar. Recall that we have $\Omega_4^{\mathrm{Pin}^c} \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$, with generators $\mathbb{RP}^4$ and $\mathbb{CP}^2$. Thus, the q-component is determined as in the Type III case. The s-component of P is determined by the bordism number $\langle w_2(P)^2, [P]\rangle \in \mathbb{Z}/2$. (Here we use the notation given before Theorem 3.6.) Since $\mathrm{KS}(X) = 0$, there exists an integer m such that $X_0 = X\sharp m(S^2 \times S^2)$ is smooth. Note that if we do the same construction on X_0, we get $P_0 = P\sharp m(S^2 \times S^2)$, and $\langle w_2(P_0)^2, [P_0]\rangle = \langle w_2(P)^2, [P]\rangle$. Therefore, to compute the s-component, we may assume that X is smooth. Recall that $P = (X - \mathring{N}(F)) \cup_\partial B$; it is seen that the bordism class of P is determined by the bordism class of the pair (X, F), which can be viewed as a singular manifold $(X, f) \in \Omega_4(BU(1)) \cong \Omega_4 \oplus H_4(BU(1))$. We have two homomorphisms

$$\Omega_4(BU(1)) \longrightarrow \mathbb{Z}/2, \quad [X, F] \mapsto \langle w_2(P)^2, [P]\rangle$$

and

$$\Omega_4(BU(1)) \longrightarrow \mathbb{Z}/2, \quad [X, c_1(\widetilde{\xi})] \mapsto \langle w_2(X)^2 + c_1(\widetilde{\xi})^2, [X]\rangle.$$

By a check on the generators $(\mathbb{CP}^2\sharp(S^2 \times S^2), c_1(\tilde{\xi}) = (1, 0, 1))$ and $(\mathbb{CP}^2\sharp(S^2 \times S^2), c_1(\tilde{\xi}) = (0, 0, 1))$, we see that $s = \langle w_2(P)^2, [P]\rangle = \langle w_2(X)^2 + c_1(\tilde{\xi})^2, [X]\rangle \pmod 2$. The two cases correspond to the values $s = 0$ and $s = 1$. For the proof of (2), the only change is that $\Omega_4^{\mathrm{Top}} \cong \mathbb{Z} \oplus \mathbb{Z}/2$ with generators $\mathbb{CP}^2$ and $*\mathbb{CP}^2\sharp\overline{\mathbb{CP}^2}$ [10].

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